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LXV. *The Reaction consequent upon the Evaporation of a Liquid and upon the Emission of Vapours from Small Orifices.*—PART I. *Argument.* PART II. *Appendix.*  
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INTRODUCTION.

THE object of this communication is to draw attention to a type of interference between the molecules of a vapour and those of an external medium, which appears to play a considerable part in the phenomenon of evaporation and in the discharge of a vapour from a boiler.

It deals also with the manner in which a vapour is discharged into the atmosphere from a boiling liquid and with the reaction consequent upon the discharge of molecules from a boiler maintained at a steady temperature and pressure, and with certain phenomena associated with saturated vapours which have to be considered in the light of the existence of molecular interference.

The paper is divided into two parts:—the first sketches the main features of the investigation and describes the results; this we have termed the *Argument*; the second, which consists of an *Appendix* divided into several Sections, comprises fuller details of the experiments described in the first part, together with a certain amount of mathematical treatment, whose inclusion in the outline of the dissertation would tend to obscure the trend of the discussion.

\* Communicated by the Author.

## PART I.

## THE ARGUMENT.

*The Origin of the Investigation.*

IN the course of a series of experiments upon an electric arc burning in air, it was discovered that each pole behaved as though there were a mechanical pressure upon it which caused it to move away from the opposite pole\*. It is not necessary here to discuss this phenomenon, it is sufficient to state that various explanations of this pressure suggested themselves, amongst others that it was the reaction consequent upon the evaporation of carbon atoms from the hot poles. It was the necessity for testing this experimentally which led to the present investigation; the results of some of the earlier experiments have been briefly stated in the papers to which reference has been made. A somewhat similar problem had previously confronted Osborne Reynolds, who gave reasons, stated in Section 10 (Appendix), for expecting a reaction upon an evaporating surface.

*Evidence of Interference in the Process of Evaporation.*

The investigation has been carried much further than the original problem demanded, which was merely to discover whether in the process of evaporation or boiling into the open air, there were any measurable mechanical reaction upon the surface from which the molecules were issuing. This was, however, the first matter submitted to experimental test. Water and ether were the substances found most suitable and, briefly, it may be stated that no measurable reaction was observed when these evaporated into the open air. This experimental fact led to an examination of the reasons why it is absent, or at least too small to be measured.

There is a real and, I believe, a new problem here; for suppose that there is an escape of molecules of water vapour at the rate of something like  $\cdot 03$  gm. per second (as in an actual experiment), and that the molecules of steam are escaping from the liquid with a speed which is of the order of magnitude associated with a gas molecule at the temperature of the experiment (roughly  $7 \times 10^4$  cm. per second for steam at  $100^\circ$  C.). This is, I believe, the generally, but not universally, accepted view; it was adopted by O. Reynolds (*loc. cit.*), and experiments by Knudsen seem to support it.

\* Duffield, Burnham, and Davis, Phil. Trans. Roy. Soc. A. 220, p. 109 (1919). Proc. Roy. Soc. A. 97, p. 326 (1920).



The matter is further considered in Section 4, Appendix, but we adopt it here provisionally as a working hypothesis. On this basis we see that the surface should be losing momentum at the rate of  $\cdot 03 \times 7 \times 10^4 = 2100$  dynes, which is nearly equal to the weight of 2.14 gms. This is, therefore, the order of magnitude of the force to be expected if the molecular projections were the only factor to be taken into account; if only the fastest molecules were projected the rate of loss of momentum would be still greater; on the other hand, if the projection were at random instead of normal to the surface, we should divide by  $\sqrt{3}$ .

In another experiment, the evaporation proceeded so rapidly that the rate at which the liquid lost momentum, on account of the projection of molecules, corresponded to the weight of 11.4 gms. Obviously no very refined experiment should be required to detect so large a reaction, and that is why we have been content in the first instance with a rough experimental investigation whose object was to detect whether or not there is a reaction of this order of magnitude.

The first experiment was conducted by allowing ether to evaporate from the pan of a balance and observing the rate at which the pointer moved (Section 1)\*. Subsequently (Section 2), it was shown that it was legitimate to experiment with a liquid which was boiling, because it was found that a boiling liquid does not in the main, or even largely, discharge its vapour through the bursting of bubbles, but by the passage of the material through the bubble-wall after the bubble has risen to the surface; so that in its final stage the phenomenon of boiling is closely allied to that of evaporation in that the essential action takes place at the surface.

A second set of experiments was therefore carried out in which a liquid was boiled upon the pan of a balance (Section 3). The writer would be the last to claim that these experiments are of a refined character, but they certainly appear to be adequate to determine whether the molecular projection was the only factor involved in the process of evaporation and boiling into the open air.

In none of the experiments was any pressure upon the surface observed, and we are well within the limits of experimental errors when we say that we could easily have detected a reaction equal to the hundredth part of that whose existence we wished to test.

\* The reference is to Section I of the Appendix forming Part II. of this paper. Subsequent references will be similarly abbreviated.

The problem is to account for the absence of any measurable reaction.

Is it absent because (1) the general view is wrong that the molecules issue with the velocity we have assigned to them; do they, instead of being projected like bullets, just drop off? Do they suffer so much in velocity in penetrating the surface layer? Do they, in fact, behave like electrons liberated photoelectrically from a metal at the threshold frequency? We devote Section 4, Appendix, to a consideration of this point, and as the evidence obtained *in vacuo* points to a high velocity of projection, it seems that this is not the explanation, so we proceed to consider the other alternative (2), namely, that there is some compensating interaction between the projected molecules and the molecules of air or previously evaporated material in the space above the liquid, which prevents the loss of momentum of the projected material from becoming evident as a pressure upon the surface?

The gist of the experimental evidence, though not absolutely conclusive, is that the evaporative reaction *in vacuo* is powerful and in air imperceptible, hence we conclude that the explanation lies in this latter alternative, and what we must next proceed to do is to examine the effect upon the motions of the molecules of air or vapour above the liquid of the projection among them of a large number of molecules moving with high speeds from the liquid below. We must observe two precautions in dealing with the problem—it would be misleading and erroneous to treat the problem as though the evaporated material were introduced into the atmosphere as an ordinary mass of gas with the usual Maxwellian velocity distribution; this is not the case, since the molecular motions are all in the forward direction though precisely how distributed about the normal to the surface is not at present known. Moreover, the permanently escaping material is not to be treated as saturated vapour; this we may see by comparing the number of molecules which would cross unit plane within a saturated vapour with the number actually lost by the same area of a liquid surface under vigorous boiling. As we show later, the ratio is not likely to be much less than 5000 to 1. Or we may put the matter thus:—The quantity of vapour crossing unit plane in a saturated vapour in a second at atmospheric pressure is obtained by writing  $\Pi = 10^6 = 2mnv$ , where  $n$  is the number of molecules crossing unit plane in one second with velocity  $v$  normal to it, whence for steam at  $100^\circ$  C. the mass  $mn = 10^6/2v = 10^6/2 \times 6 \cdot 6 \times 10^4 = 7 \cdot 6$  gms.



This is enormously greater than the greatest rate of boiling attained, namely  $\cdot 0015$  gm. per sec. per unit area.

It is true of course that the newly evaporated molecules will, after projection, find themselves amongst previously evaporated material, and that the whole space may be saturated, but the point is that the material as it issues is far from possessing the density of saturated vapour; we might indeed imagine so slow a supply of heat that only one molecule issued from each sq. cm. in a second. It should be clear from this that the ordinary equations relating to the mass-motion of a vapour are not applicable to this problem; we proceed therefore from another point of view.

If the external molecules which are set in motion by collision with the projected material were originally stagnant, at rest, the momentum of the whole mass would still be that of the projected material. Imagine a platform on which a thousand cannon were mounted, all pointing in the same horizontal direction, and each of which fired a cannon-ball per second. The reaction upon the platform would be the same, whether the space before it were empty or were occupied by a large number of similar balls suspended by cords which broke when the balls were struck. The loss of momentum per second by the platform would be the same, but there would be a different number of balls in motion in the two cases—in the first a thousand would be set in very rapid motion in each second, whereas in the latter there would be many thousand moving with much smaller speeds, but representing the same total momentum away from the platform. The full recoil would be felt by the platform in each case.

Clearly, since in our actual experiments an evaporating surface remains sensibly at atmospheric pressure even when discharging molecules with enormous velocities, there must be a further factor to be taken into account.

It was suggested that the condensation of the vapour into water above the liquid might occasion a reduction in the pressure which would just balance matters. This was tested, not by the rather insensitive balance method, but by using a jet of steam and observing the reaction upon the vessel from which it was issuing by a delicate torsion method, both when the steam was condensed upon a cold body placed close to the orifice, and also when a body at a temperature over  $100^{\circ}$  C. was substituted. Though in one case the condensation was vigorous and in the other was absent, no difference in the reaction was observed. The areas of the hot and cold bodies were 100,000 times that of the orifice, and they were

placed in turn at distances varying from 7 cms. to 0.2 cm. from the orifice. This is strong evidence that condensation, as usually understood, is not the cause of the phenomenon.

What then is the further factor to be taken into account? In the writer's opinion it is that the molecules of the external medium are in motion and that this is profoundly affected by the projected material.

The liquid is being continually bombarded by molecules of air or vapour whose impacts occasion the usual gas pressure, but, as the rate of evaporation increases, some or all of these suffer encounters with the molecules projected from the liquid, so that they are prevented from communicating their full momentum to the surface.

This may happen either through the deflexion of the molecule or the reduction of its velocity normal to the surface as the result of a collision or series of collisions. The precise mechanism whereby the momentum is diverted or neutralized we do not propose to examine here; we should not, however, dismiss lightly the possibility of some collisions being of an inelastic character, because Wood has shown from spectroscopic evidence that a newly evaporated molecule of mercury may, instead of condensing, attach itself to a molecule of hydrogen in the space above the boiling liquid, and there is also in a vapour always the possibility of molecular association. Also we find Langmuir urging the inelastic nature of the encounters close to an evaporating solid surface.

Our view thus is that there is a type of interference between the rising molecules and those directed downwards from above, and the reason why the pressure upon the surface is not the simple sum of the atmospheric pressure and that due to the recoil is that the downward stream fails to impart to the surface the momentum which has been deflected or otherwise absorbed.

It is as though our gun-platform were being bombarded by enemy fire, which we can picture as occasioning a pressure which might be measured by the compression of a spring. Upon our opening fire, there is an additional reaction if our cannon balls do not strike those of the enemy (*i. e.* if the cannon ball density is low), but if they do collide in such a way that each ball stops or diverts an enemy ball, the reading of the spring will remain unaffected by the discharge of our guns, since each discharge both contributes a certain amount of kick to the platform and robs it of the impact of an enemy shell.

But it is not infinitely probable that each enemy shell will



meet one of ours in such a way as to be prevented from delivering its momentum to the platform, so the neutralization is not likely to be complete unless the discharge rate is very great. The amount of momentum stopped depends, for example, upon the size of the shot and the number fired in each second. This then is much what is happening in the case of a liquid which is evaporating or boiling—it depends upon the rate of evaporation how much of the external atmosphere is prevented from exerting its pressure upon the surface.

In the experiments we have described the number of molecules escaping per second was always a very small fraction of those incident upon the surface in that time:—for example, for a moderately rapid rate of boiling, 0.0015 gm., or  $5 \times 10^{19}$  molecules, escaped per second from each sq. cm., whereas the number incident from above was approximately  $3.4 \times 10^{23}$  in the same time, so that the number escaping was less than the 1/5000th part of those with velocities directed towards the surface. In no case was this rate of evaporation greatly exceeded, so to test our theory it becomes increasingly important to find other means whereby we may achieve a greater ratio between those emitted and those approaching the surface from without.

Before discussing these, there is one simple case of some theoretical interest—suppose that even with small discharges all emitted molecules were to meet external molecules in effective encounters; let us gradually increase the rate of discharge; we should expect to find no reaction upon the liquid so long as there were molecules in the external medium with velocities directed towards the surface, which could be stopped or diverted by those emitted. When the latter are all met compensation ceases, and any additional molecule projected from the liquid will then contribute a reaction which cannot be balanced by any interaction with the atmospheric molecules, so a pressure in excess of that of the atmosphere will be experienced. It is easy to see, or if there is any difficulty, Section 8, which will be referred to later in the Argument, will make it clear, that the limiting rate of discharge without reaction occurs when twice as many molecules leave the surface as would have struck it from above if there had been no evaporation; a simple calculation (*loc. cit.*) indicates that if the liquid and the vapour above are both sensibly at 100° C., the limiting rate is approximately 15 gms. per second per sq. cm., so, for the particular case we have imagined, we should expect zero reaction below this rate and a rapidly increasing reaction

above it. We shall see later that there is some indirect experimental evidence in accord with the figure we have obtained.

We may now return to our search for a method of testing the interference between the emitted and external molecules. Two means suggested themselves—we may either alter the number of external molecules or we may increase the number emitted.

The former was tried first:—the theory indicated that if the external atmosphere were removed, we should obtain the full effect of the reaction due to the projection of molecules and, consequently, that we should find a measurable reaction. The experiment has already been described in Appendix, Sections 4 and 5, and as a powerful reaction was observed, it was in accord with our theory.

As this mode of attacking the problem did not lend itself readily to quantitative measurement, it seemed desirable to find a means of increasing the rate of discharge per sq. cm. of high speed molecules into the atmosphere at ordinary pressures. It occurred to the writer that the efflux of steam from a small aperture in the side of the boiler might be made to fulfil the necessary conditions. By using a large boiling surface and a small aperture the number of molecules driven out in each second per unit area may be increased enormously.

*Evidence of Interference in the Discharge of Vapour  
from an Orifice.*

If heat were supplied to a boiler at such a rate that the wall and contents always remained at a uniform temperature, it is clear that the molecules would issue into a vacuum space with velocities appropriate to the temperature. It seemed, therefore, legitimate to regard the effect of discharging into an external atmosphere as compounded of an effect due to the issue of these molecules with their temperature velocities and that due to the external medium after collision with them. It seemed very probable that across some section, not necessarily identical with the plane of the aperture and not necessarily within the boiler, it would be permissible to treat the discharge as we have already treated that from a liquid surface. Indeed, it seems reasonable to imagine that across this section, which we may designate as the "effective aperture," provided the boiler is maintained at a steady temperature and pressure, the molecules escape with the same speed as that with which



they enter the vapour space from the liquid. This point is argued in Section 6, Appendix. It marks a departure from the usually accepted idea of the efflux of vapour from an orifice, but for the case considered, namely that in which the efflux is due to the evaporation of a liquid, it is believed to be correct. It is possible that the reason why it has not been recognized is that the interference between the emitted molecules and those outside the vessel has masked the effect.

The case differs, of course, from the escape of a gas or vapour into the atmosphere from a vessel at a higher pressure, because the pressure will not then be steady, and steadiness of temperature and pressure are essential conditions, but we can attain steady conditions if we introduce molecules to the same number and with the same speeds as those which depart. This is automatically done in the case of the boiler with a hole in it, if we keep the supply of heat steady.

It is almost as though we could regard each molecule introduced into the vapour space as a little piston moving with the molecular velocity whose effect would be to drive one molecule out from the "effective aperture" with the same speed.

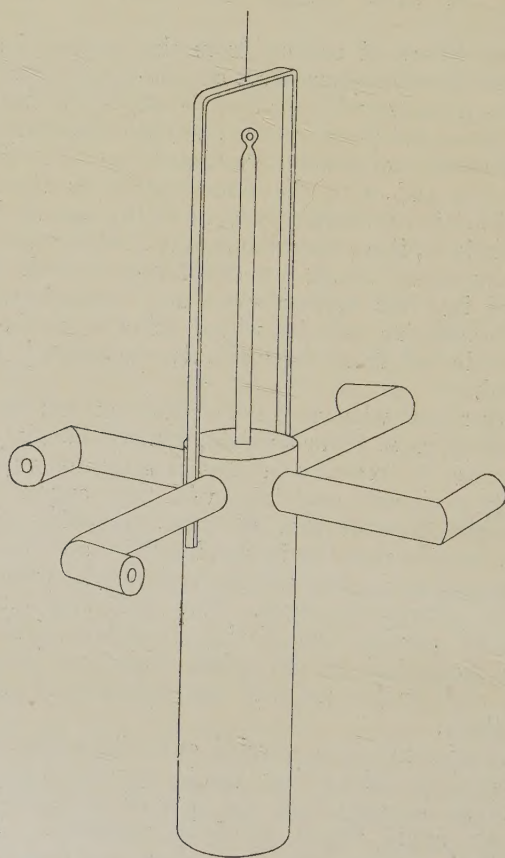
We may repeat what we have already said concerning the material evaporated from a liquid, namely that the escaping vapour cannot be treated as a mass of saturated vapour with the density ordinarily associated with that temperature.

The justification of this and the preceding theory of molecular interference lies in its simplicity and its ability to give a reasonable explanation of experimental results, as we shall now attempt to show. We shall first outline the experimental results, then derive a mathematical formula from the theoretical considerations which we have already adduced, and finally indicate the measure of agreement between them.

The experiments were carried out with a form of Hero's engine, consisting of a boiler (fig. 1) suspended by a torsion fibre, and so disposed that the issue of steam twisted this through an angle which could be measured. The total reaction,  $R$ , in excess of the atmosphere, was thus calculable. The apertures at the ends of the arms were circular holes in thin brass sheet; their size and number could be varied. Further details of the apparatus and method of experimenting are relegated to Section 7, it is sufficient for the purpose of the Argument to indicate the relationship between  $R$  and  $D$ , the quantity of steam discharged per second, for apertures of different diameters.

When the aperture was as large as the undisturbed surface of the boiling liquid, it was not possible to obtain a rate of discharge which would occasion any measurable recoil—this appears to be strong evidence of the existence of interference between the issuing and external material: in one

Fig. 1.



experiment the rate at which the boiler lost momentum on account of the projection of molecules would have occasioned a twist of 30,000 degrees if there had been no compensating interaction with the surrounding medium, or if it had been discharging into a vacuous space, but no twist was observed.



Other experiments with less sensitive suspensions are quoted in Table I. to the same effect.

TABLE I.

The Reaction observed with large apertures.

Observers: W. G. Duffield & J. S. Burgess.

Ratio of area of apertures to boiling surface area.	Mass evaporated per second, D gms.	Deflexion observed.	Deflexion which would have been observed if there had been no interference.
1	0.016	-5°0	6300°
	0.077	0°	32000°
	0.026	0°	9510°
	0.086	+4°0	30200°
.75	0.044	0°	16300°
	0.018	0°	810°
.5	0.034	+1°8	12300°
	0.024	2°0	1060°
	0.040	2°0	1770°
	0.057	2°2	20700°
.25	0.032	1°6	11600°
	0.007	4°0	2580°

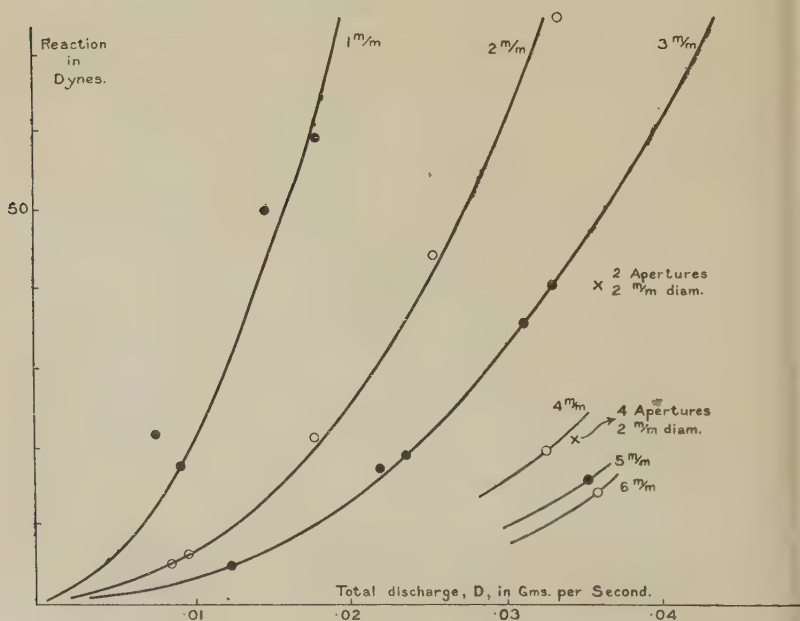
Torsion wires of varying sensitivities were used in different experiments.

Upon diminishing the aperture area a small reaction was observed which increased slightly as the aperture was further reduced in size; this is shown by the latter half of Table I., but the reaction is still almost negligibly small in comparison with the effect which would have been produced if the atmosphere had been stagnant.

*Further Experiments upon the Discharge of Vapour from Small Orifices.*

The results of experiments with still smaller apertures are given in Table III., Section 7, from which it will be seen that reactions of the order of some hundreds of dynes were obtained; it is sufficient here to indicate the results graphically, as in fig. 2, where  $D$  and  $R$  are plotted. The interesting fact emerges that for a given discharge rate the reaction is

Fig. 2.

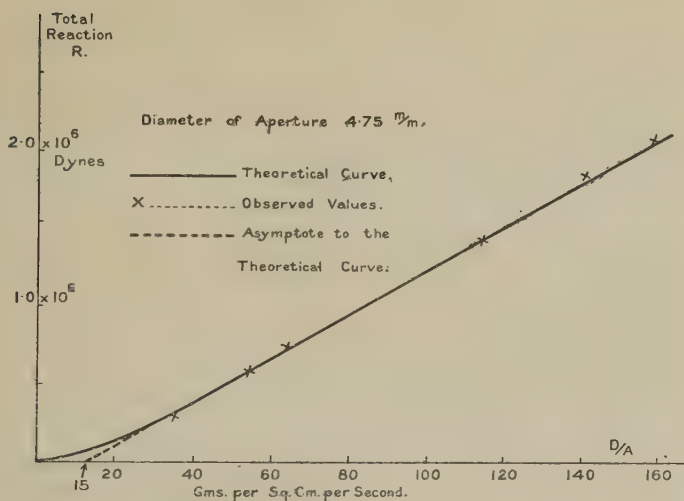


greatest when the aperture is smallest. The effect of doubling the number of apertures is approximately to halve the reaction. We note the rate at which the reaction varies with the discharge; at first it is approximately parabolic but subsequently becomes more nearly linear. The character of the graph for very high rates is best shown by fig. 3, which represents the results of a series of observations by Rosenhain with an orifice 4.75 mm. in diam. (*vide* Section 7); here, however, we have plotted the reaction,  $R$ , against  $D/A$ ,



the discharge per unit area per second, instead of  $D$ , as in the previous figure. Disregarding the theoretical curve, which will be discussed later, we note that the graph shows a nearly linear relationship as high values of  $D/A$  are reached.

Fig. 3.



### *Theoretical Investigation of the Discharge from Orifices upon the Interference Hypothesis.*

Let us now see if our theory can account for the shapes of these curves. In dealing with this problem, which is really a statistical phenomenon of considerable complexity, we shall make certain assumptions which will greatly simplify the mathematical treatment; they should at least provide us with the *form* of equation relating the reaction to the discharge. For example, we suppose all molecules to possess the average velocity associated with their temperature on the kinetic theory, and we also suppose that interference between two streams of molecules, instead of reducing the velocities of a large number of them, completely stops a certain fraction of each stream, the remainder proceeding with undiminished velocities; these simplifications would probably affect the values of the constants in our final

equation if we were to attempt to evaluate them from probability considerations, but, as we shall determine their value from the data of a particular experiment, it is probable that our results do not suffer.

Our view of the phenomenon of the discharge is as follows :—

From the “effective aperture” a certain number of vapour molecules  $n_i$  are discharged in each second, a certain number enter in each second, and  $L$  is the excess of those emitted over those which enter in that time. These  $n_i$  molecules leave with an average velocity appropriate to the temperature, as already explained (Section 6), of which  $v_i$  is the average component normal to the orifice. Their distribution about the normal to the orifice is unknown, it may be that it is a Maxwellian distribution in the forward direction only, or the projection except very close to the edges may be normal.

There will be a reaction upon the vessel due to these, amounting to  $mn_i v_i$ .

The other source of pressure upon the vessel arises from its bombardment by molecules from the atmosphere, which we may assume to be molecules of water-vapour. These, if there were no molecular interference, would contribute a pressure to each unit area of the effective aperture equal to the rate at which they deliver momentum to it, but the fact that collisions will occur between these molecules and the projected molecules occasions a reduction in the momentum which reaches it from the external medium. An important part of our problem is to find out how much of this atmospheric momentum is prevented from reaching the vessel, and how much gets through to it; we shall, for the present, denote by  $\beta$  the fraction which is prevented from reaching unit area of the effective aperture. We shall see later that it is likely to be a simple function of the rate of discharge of vapour from the vessel. The conditions near an evaporating surface and near an orifice are set forth in Sections 8 and 9. The problems are regarded as identical, except that the question of reflexion does not arise in the case of the orifice, and that simplifies matters. The orifice problem is free also from the possibility, which arises when using a horizontal discharging surface, of some condensed vapour falling back upon the liquid. We may note that if this did occur in the evaporation experiments, which the



writer doubts, it would mean that there was more interference than was supposed, because the original discharge would have been underestimated.

In the case of the discharge from an orifice the formula (see Section 8, Equation 11) for the total reaction  $R$  upon a vessel from which steam is issuing at the rate of  $D$  gms. per second from an aperture of area  $A$  is given by

$$R = Dv_i + \frac{\Pi A}{2}(r-1) - \Pi A \frac{r+1}{2}\beta,$$

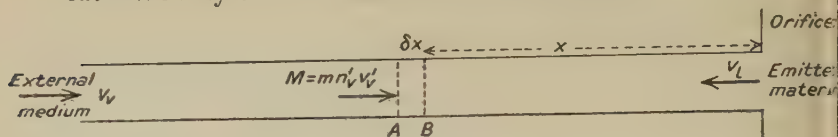
where  $r$  is the ratio of the average velocity  $v_i$  of the discharge in the normal direction, to  $v_v$ , the velocity with which an external molecule strikes the surface. The assumption has been made that interference operates by reducing the number of external molecules which penetrate to the aperture rather than by reducing their velocity.

If we wish to compare this theoretical result with those obtained experimentally, we require further information regarding the value of  $\beta$ .

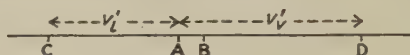
The simplest way of examining the nature of the function  $\beta$  is to look upon the ejected material as "absorbing" the momentum of those molecules of the external medium which would have entered the vessel if there had been no discharge. We do not know precisely how the "absorption" takes place, whether the external molecules are merely diverted from the orifice, or whether their momentum is partially neutralized by non-elastic collision with the issuing molecules. We may obtain the *form* of the equation relating the loss of momentum to the rate of discharge of the material by considering a simple case:—

Assume that the jet is parallel and that condensation does not occur in the range under discussion; assume also that all the issuing molecules move in a direction normal to the aperture, and that the pressure of the external atmosphere is due to a stream of molecules directed along the axis of the jet, each molecule being assumed to possess the average velocity of the stream. There are thus two opposing streams of which the former tends to neutralize or absorb the momentum of the latter. Let the jet be of unit cross-section, and let  $M$  be the total momentum of all the particles of the external medium which penetrate to a distance  $x$  from the orifice in each second. In traversing a further distance  $\delta x$ , there will be a further loss or absorption of momentum,  $\delta m$ , in each second. We should expect this to be proportional

to  $M$  and to the number of emitted molecules in the volume  $\delta x$ . We may examine the matter as follows:—



Let  $n_v'$  be the number of molecules of the external medium which cross the section A, distant  $x$  from the orifice, in each second. Let their average velocity at this point be  $v_v'$ . We require in the first instance to know the total number of collisions made by these  $n_v'$  molecules in traversing  $\delta x$ . We shall first find the probable number of encounters with the emitted molecules which a plane area at A would suffer if it were moved forward to B:—The molecules of



emitted material which are at A at the beginning of a particular second would, if their velocity were maintained, be at C at the end of that interval, where  $AC = v_l'$ , the velocity of that material at the point  $x$ . In the meantime the plane would have moved from A to D, a distance  $v_v'$ , and so would have encountered all the molecules contained in the length CD in one second, where  $CD = v_l' + v_v'$ . But the time taken by the plane to go from A to B is  $\delta x/v_v'$ , consequently, in going this distance the plane sweeps through a column of molecules whose length is  $(v_l' + v_v')\delta x/v_v'$ .

Let there be  $N_x$  projected molecules in each unit length near the point  $x$  at any instant, then the number of molecules encountered by the unit plane is

$$N_x(v_l' + v_v')\delta x/v_v'.$$

But the effective area of the  $n_v'$  molecules which cross plane A in one second is  $\pi\sigma^2n_v'$ , where  $\sigma$  is the molecular diameter, so the probable number of collisions between the two streams in the region  $\delta x$  in one second is

$$\gamma\pi\sigma^2n_v'N_x(v_l' + v_v')\delta x/v_v',$$

where  $\gamma$  is a constant. Of these a certain fraction  $\lambda$  may be classed as effective in absorbing the momentum of the external medium.

$\lambda$  is possibly a function of the relative velocities of the molecules in the two streams, but there are two reasons for



regarding it as approximately a constant, (1) because Jeans found that the expectation of the velocity of an individual molecule after an elastic encounter being in the original direction varied only from .33 to .5, though the relative velocities varied from 0 to  $\infty$ , and (2) because the case which we shall follow most closely is that in which the absorption of momentum is due to a diminution in the number of molecules reaching the surface, the velocities of the remainder being unaffected, though a similar expression results from a consideration of a general reduction in the molecular velocities. On this basis we see that each effective collision deprives the oncoming stream of  $mn_v v'$  units of momentum; consequently the amount of momentum absorbed is given by

$$\delta M = \lambda \gamma \pi \sigma^2 m n_v' v_v' (v_l' + v_v') N_x \delta x / v_v,$$

which, since  $M = mn_v' v_v'$ , and  $v_l' = v_l$  and  $v_v' = v_v$ , becomes

$$\delta M = \lambda' M N_x \delta x (v_l + v_v) / v_v,$$

where  $\lambda'$  is another constant. Writing  $v_l/v_v = r$ , as before, we have

$$\delta M / M = \lambda' (r + 1) N_x \delta x.$$

We may integrate this from the region  $x = v_l$ , where we should have  $M = mn_v v_v$  to the aperture where  $x = 0$ , and  $M$  has the value  $M_0$ , thus:—

$$\int_{M=M_0}^{M=mn_v v_v} \frac{dM}{M} = \lambda' (r + 1) \int_{x=0}^{x=v_l} N_x dx.$$

The quantity under the integral on the right-hand side is the total number of molecules,  $n_l$ , emitted in one second, whence

$$\begin{aligned} \log M_0 - \log mn_v v_v &= -\lambda' (r + 1) n_l \\ &= -\frac{\lambda'}{m} (r + 1) mn_l = -\mu (r + 1) mn_l, \end{aligned}$$

where  $\mu$  is a constant for a given set of external conditions, hence

$$M_0 = mn_v v_v e^{-\mu(r+1)mn_l}.$$

If, instead of assuming that all the material were confined to a parallel jet normal to the aperture, we regarded it as distributed over a hemisphere, we should expect to obtain an expression of the same form but with a constant of different value.

We have previously written  $\beta$  for the fraction of the momentum  $mn_v v_v$  which was directed towards, but failed to reach the effective aperture (Section 8, Appendix), consequently

$$\beta = 1 - e^{-\mu(r+1)mn_l} \quad . \quad . \quad . \quad (13a)$$

Let us now apply this formula to the jet experiments. For reasons given in the note below, we write  $mn_l = mL = D/A$ , where, as before,  $D$  is the discharge from area  $A$ , whence

$$\beta = 1 - e^{-\mu(r+1)D/A}.$$

Substituting in equation (11), Section 8, Appendix, we have

$$R = Dv_l - \Pi A + \Pi A \frac{(r+1)}{2} e^{-\mu(r+1)D/A} \quad . \quad . \quad (14)$$

*Note.*—We have written  $L$  the excess number of molecules emitted over those which return, in place of  $n_l$ , the total number emitted; this was originally an assumption but it seems amply justified by results and is probably correct for two reasons: first, because, as we have stated in Section 6, it is doubtful in the jet experiments if any of the  $n_v(1-\beta)$  molecules actually reach the vessel, in which case the observed loss,  $D$ , is the loss of all the  $n_l$  molecules, and secondly, because in the present case we deal with what may be called "external" interference only, *i. e.* interference *additional* to that which may take place when the exchanges are equal and there is no loss of weight; the latter type of interference we may term "internal," and, as we shall see, there is some evidence that it is of importance in the process of evaporation of a liquid into its own vapour. Thus only the excess,  $L$ , will be involved in the shielding action.

In order to be consistent we have taken  $mn_v v_v$  as equal to  $\Pi/2$ , ignoring the possibility of a diminution of this quantity of momentum as it penetrates the stream of projected molecules—strictly speaking  $n_v$  is not the total number of molecules which would have crossed the aperture if there were no projection of molecules from it, but the number which would have crossed it if there were equality of exchanges, *i. e.* if  $L=0$ . This aspect of the subject is further discussed in Section 10, Appendix, and on p. 665.

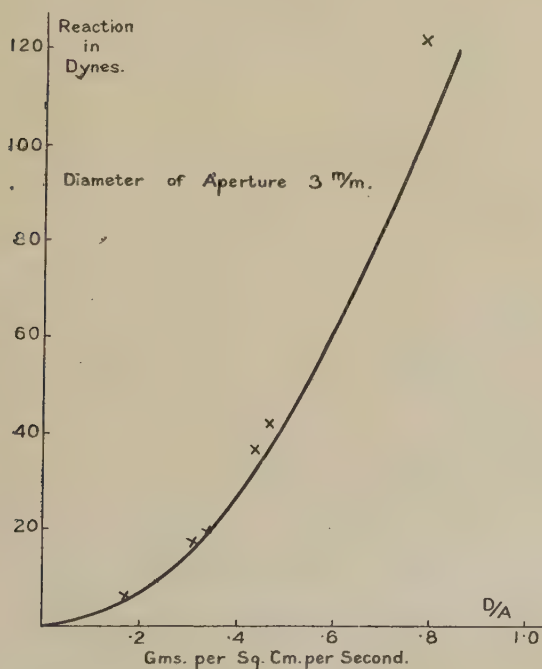
We may note that if  $L$  could not be regarded as equal to  $n_l$  in the jet experiments,  $\mu$  would vary slightly with the discharge rate and the value we obtain for it would be its limiting value when  $L=0$ . Our treatment of Internal Interference on p. 665 would not be affected.



*Application of Formulæ to Experimental Results.*

Let us now see if the equation (14) which we have obtained is in accord with our experimental results. Unfortunately the constant  $\mu$  is not obtainable from theoretical considerations, while we remain in ignorance of the precise mechanism whereby a collision is rendered effective; we require to know wherein an effective differs from an ineffective collision. We may, however, evaluate  $\mu$  from a single series of observations and test its applicability to other series. If our equation be tested this way we have very excellent evidence that it is of the right form.

Fig. 4.



From the series of observations with the .53 mm. aperture it was found that  $\mu$  was a constant and equal to 0.033. Introducing this value into the equation and solving for values of  $\bar{A}$  appropriate to the different apertures, we find the reactions plotted in figs. 4, 5, 6, and 7, full lines, for different values of  $D/\bar{A}$ . It will be seen that in all cases the observed reactions fall very approximately upon the theoretical curves, showing a very satisfactory measure of agreement

Fig. 5.

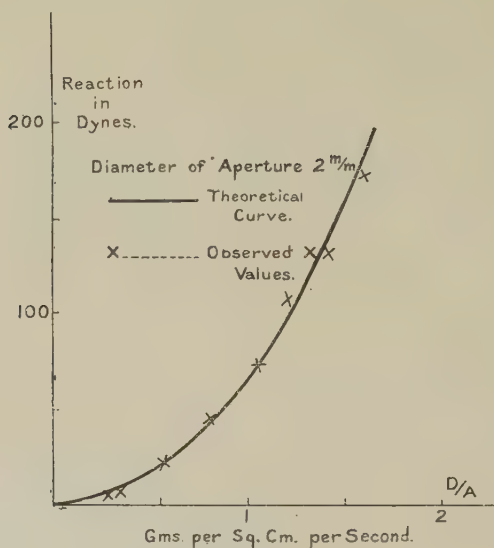
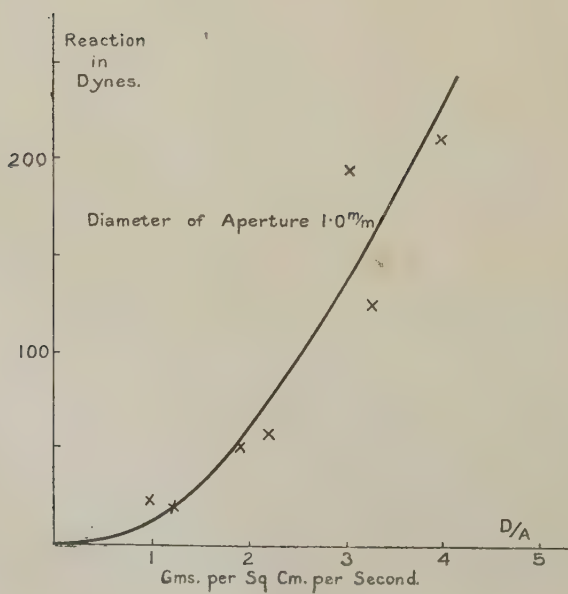


Fig. 6.

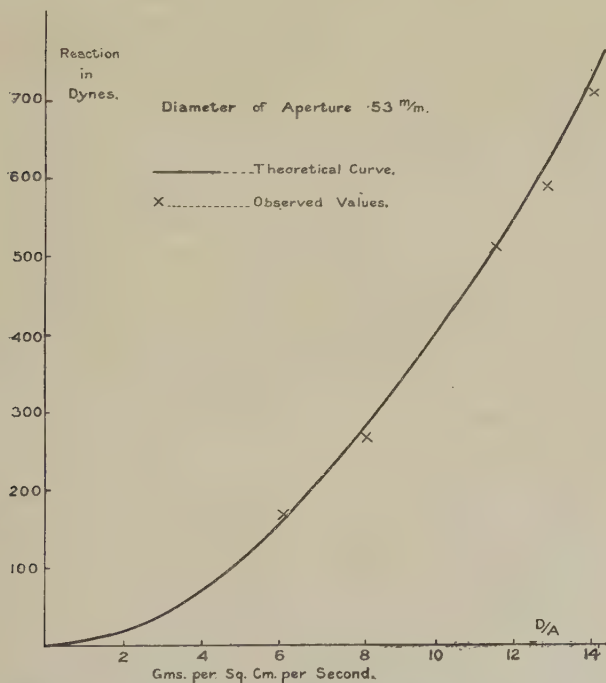




between theory and experiment. Numerical values are given in Table III., Appendix, Section 7.

The above comparison indicates that our theory satisfactorily explains the behaviour of steam issuing from an orifice under pressures varying from  $1/4000$  to  $1/3$  of an

Fig. 7.



atmosphere. We next proceed to test our equation for the high rates of discharge and high boiler pressures examined by Rosenhain. For the first two or three observations we take  $\mu=0.033$  as before, but after these the value of  $D$

becomes so large that the factor  $e^{-\mu \frac{D}{A} (r+1)}$  becomes negligible;  $\beta$  is then practically equal to unity, which means that all the external momentum is absorbed before it reaches the aperture; the curve then approximates to

$$R = Dv_l - \Pi A. \quad (15)$$

We have shown in Section 7, Table III. (note), how the value of  $v_l$  has been determined, so we may calculate the values of  $R$  for different values of  $D/A$ ; these are represented by the full line in fig. 3, the experimental values are shown by crosses, and again the agreement is very good. Thus our

equation proves satisfactory over a range of reactions varying from 6 dynes (as in fig. 4) to over  $10^6$  dynes (as in fig. 3), the corresponding rates of discharge varying from 0.2 to over 150 gms. per sq. cm. per second. Is it possible that such a measure of agreement can be accidental? \*

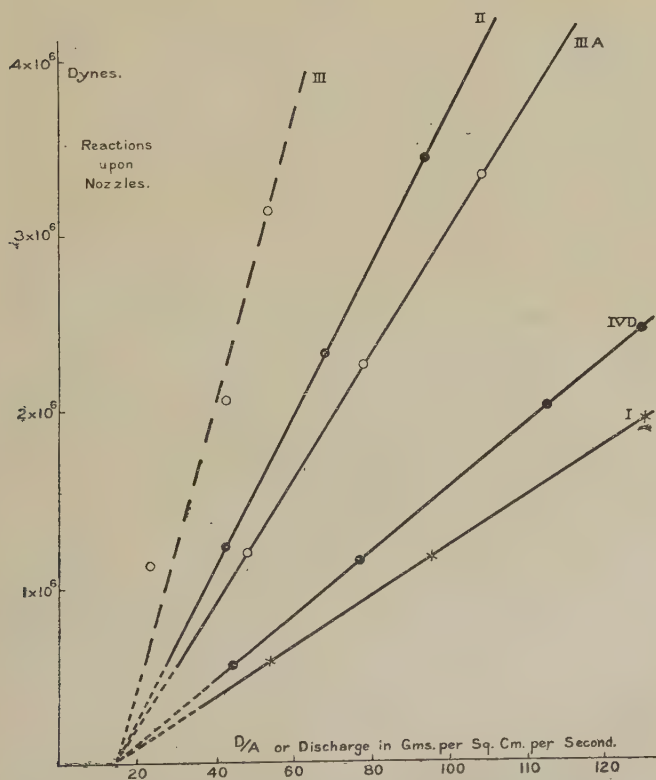
We may here draw attention to three interesting features of these researches :—(1) We saw on p. 647 that, on the supposition that a surface was discharging molecules into a vapour at the same temperature ( $r=1$ ) and that each molecule approaching the surface was effective in neutralizing the momentum of an issuing molecule ( $\beta=1$ ), it should be possible to attain a rate of discharge of approximately 15 gms. per sq. cm. per second without incurring a reaction above that of the atmospheric pressure; we saw also that this was the condition determining that twice as many molecules left the surface as would ordinarily have been incident upon it. It would be interesting to test this experimentally, but unfortunately the conditions are not capable of realization in practice, because for low discharge rates, when  $r \approx 1$ , every external molecule does not succeed in colliding effectively with one of the projected molecules, *i. e.*  $\beta$  is not unity; and also because, if we were to attain such a high rate of discharge that  $\beta=1$ , we could not do so without increasing the temperature of the interior, when  $r$  becomes greater than 1. We may, however, note that for the highest rates of discharge shown in fig. 3, for which  $\beta$  is nearly equal to unity, the points of observation lie very approximately on a straight

\* [The agreement is certainly greater than might have been expected in view of the incomplete data available for the high pressure work, in the reduction of which we have had to make certain approximations referred to in the note attached to the Table recording them (Section 7, Appendix). If we had attributed to  $v_i$  the mean value instead of the R.M.S. value of the molecular velocity, the agreement would have been less good by about 8 per cent. for the highest rate of discharge, but, as we have stated, there seemed a definite reason for regarding the mean value as being too low; how much higher to take it we did not know, and we ought not to claim more than that it was a good guess to have taken it as the R.M.S. value, since that makes the agreement practically perfect over the whole range. We may note, however, that the matters referred to in the next two paragraphs are independent of the precise value assigned to  $v_i$  in the high pressure work.]

We had not hoped to show more than that the general phenomenon of evaporation and discharge was of the nature we have described—there must be other factors to be superposed upon our main theory. We should, for example, expect a small effect due to the priming of the steam, to the flow of vapour near the edge of the orifice, and perhaps to the presence of associated molecules in the escaping vapour. But it would seem that these effects are small compared with those which constitute the main lines of the Argument.]

line, as we should expect from equation (12), Section 8, since  $v_i$  varies but slightly. If we produce this line, which represents the condition  $\beta=1$ , backwards until it cuts the horizontal axis, we arrive at the point for which  $\beta=1$ ,  $r=1$ , and  $R=0$ , and consequently realize, by extrapolation, the conditions we have imagined. For this point we note that  $D/A$  is very approximately equal to 15 gms. per sq. cm. per second, in excellent agreement with our simple theoretical conclusion.

Fig. 8.



It is interesting to find that not only does the aperture in a thin plate give a curve whose asymptote cuts the horizontal axis at  $D/A=15$ , but also that nozzles of various lengths, diameters, and tapers agree in giving this value to a very considerable degree of accuracy. Particulars of the nozzles used by Rosenhain and the reactions experienced at different discharge rates are given in Table IV., Section 7. The results are plotted in fig. 8. We are only concerned with



those portions of the curves for which  $\beta=1$ , *i. e.* with those parts which are appreciably straight. Drawing the asymptotes we find two things: first, that all, with the possible exception of that for nozzle III., converge towards a point on the axis; and secondly, that this point corresponds to  $D/A=15$  gms. per sq. cm. per second. In view of the fact that the nozzles possessed tapers varying from 1 in 30 to 1 in 12, and areas which varied from 0.177 sq. cm. to four times that area, the result is a remarkable one. In the writer's opinion it would be difficult to account for it on any theory other than that outlined in the preceding pages. In other respects, however, our formula does not apply to expanding jets.

(2) The second point of interest is this:—We can easily show that the asymptote to the theoretical curve in fig. 3 produced backwards cuts the vertical axis at a point corresponding to  $-10^6$  dynes per sq. cm. excess pressure, which is the same as zero, absolute, pressure. Consequently, points on the asymptote represent relationships which would have been obtained if the external medium exerted no pressure. Since the theoretical curve is experimentally indistinguishable from its asymptote a little before the first experimental point (which was obtained with a boiler pressure of 20 lb. per sq. inch) it appears that above this boiler pressure the rate of discharge is independent of the external pressure. It is thus satisfactory to find that our theory is in agreement with the well-recognized fact that the rate of discharge is independent of the external pressure when that pressure is less than half the internal pressure.

(3) We have pointed out that when the discharge rates are very small the effect of doubling the area of the aperture is to diminish the reaction by approximately one-half (*cf.* p. 652) for a constant value of the quantity discharged per second. This point is not an obvious deduction from equation (14) in its present form, but if we expand the exponential function, we may write it thus:

$$R = Dv_i - 2\mu\Pi D + 2\mu^2\Pi D^2/A$$

if we confine our attention to small discharge rates for which  $r$  does not differ appreciably from unity. Since  $v_i = 6.6 \times 10^4$  and  $2\mu\Pi = 0.66 \times 10^6$ , we see that the first two terms on the right-hand side destroy one another, leaving

$$R = 2\mu^2\Pi D^2/A, \quad . \quad . \quad . \quad . \quad (16)$$

whence  $R$  is inversely proportional to the area, which is in accord with our observations. This, as we shall see by the

next Section, will hold good as long as the discharge is very small and takes place into a dense medium, and will be true for all vapours.

### Internal Interference\*.

Let us see if we can apply the foregoing investigation to obtain more detailed knowledge of the conditions governing the discharge of vapour from a liquid during the process of evaporation.

Consider the case of a saturated vapour above the mercury in a barometer tube. The vapour pressure is measured by the depression of the mercury below its normal level and is made up of the reaction due to the evaporation of molecules from the small stratum of liquid on the top of the mercury column and the impact of molecules from the vapour space. Unless the molecules are infinitesimal there is the probability of collisions between the two streams, so that all those with velocities directed downwards, which would have struck the liquid in the absence of evaporation, will not actually reach it, or, if they do, will impinge upon it with diminished velocity.

We have previously directed our attention chiefly to the effect of this interference upon the pressure upon the liquid—let us now consider the proportion of molecules in the vapour space which are prevented from reaching the liquid under conditions of saturation.

Since there is no loss of weight of the liquid, we have, in our previous notation, but writing  $\beta_i$  for the *internal* interference coefficient, and taking  $n_v$  as the number reaching the surface in the absence of *all* molecular projection from it:

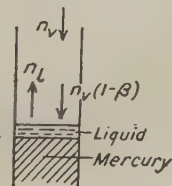
$$L = n_l - n_v(1 - \beta_i) = 0. \quad (16 a)$$

Also the vapour pressure,  $R_v$ , is the pressure upon the surface and is given by

$$\begin{aligned} R_v &= mn_l v_l + mn_v(1 - \beta_i) v_v \\ &= mn_v(v + v_v)(1 - \beta_i), \text{ or since } v_l/v_v = r, \\ &= mn_v v_v(r + 1)(1 - \beta_i). \end{aligned} \quad (17)$$

We saw from equation (13 a) p. 658 that

$$\beta_i = 1 - e^{-\mu(r+1)mn_l},$$



whence, from (16 a)

$$\beta_i = 1 - e^{-\mu(r+1)mn_v(1-\beta_i)},$$

or, using (17),

$$\beta_i = 1 - e^{-\frac{\mu R_v}{v_v}} \quad (18)$$

\* For definition, see Note, p. 658.

From our experiments with the suspended boiler we found the quantity  $\mu$  for water vapour at atmospheric pressure to be  $\cdot 033$ . Since the temperature was  $100^{\circ}\text{C}$ .,  $v_v = 6\cdot 6 \times 10^4$  cm./sec. and as  $R_v = 1\cdot 01 \times 10^6$  dynes, it follows that  $\frac{\mu R_v}{v_v} = \cdot 50$ , whence  $\beta_i = \cdot 393$ . This means that at ordinary pressures  $\cdot 393$  of the momentum directed downwards is prevented from reaching the liquid.

A number of experiments were devised to measure  $\mu$  for different liquids at their boiling-points, the apparatus already described and shown in fig. 1 being used. Electrical heating was substituted for the flame as some fumes were inflammable, and to avoid prolonged inhalation, the experiments were conducted within a draught chamber. The greater number were carried out by Mr. J. Webber, B.Sc. The results obtained are shown in Table II. The liquids used were carbon tetrachloride, ether, and alcohol in addition to water, which had previously been examined.

The value of  $\mu$  was calculated as in the case of water vapour, and again we find a remarkable agreement in the values obtained for any particular substance for all rates of discharge. More remarkable, however, is the fact that at the same atmospheric pressure the values of  $\frac{\mu R_v}{v_v}$  are practically the same for all these substances in spite of the differences in  $v_v$  consequent upon the different temperatures at which they boiled. The liquids had been chosen largely on account of their widely divergent molecular weights, which varied from 18 for water to 153\cdot 8 for carbon tetrachloride.

This discovery led to a suspicion that it might be a purely geometric result consequent upon the approximation of the experimental curve to a parabola; if these were parabolic throughout it might be substantiated, but the suspicion is dispelled by the fact that the curves are only parabolic near the origin; they soon definitely depart from this condition and become nearly linear, *vide* the points of observation in figs. 3 and 8. On the other hand, our formula is satisfied over the whole of the graph. It is more probable that the invariability of  $\mu R_v/v$  is the property of a liquid which governs the conditions whereby it remains in equilibrium with its saturated vapour, or, in other words, it enables us to define the boiling-point in terms of the numbers of molecules directed towards and away from the surface.



TABLE II.

## I. Experiments at Atmospheric Pressure.

Area of Aperture .00754 sq. cm. Diameter .98 mm.

Atm. Pressure dyms.	Discharge per second.		Temp. of Vapour.	Molec. Velocity.	Ratio $v_l/v_v$ .	Re- action obsd.			Mean value of $\frac{\mu R_v}{v}$ .	$\beta$
$R_v$ .	D.	D/A.	$T_{Abs.}$	$v_l$ .	$r$ .		$2\mu$ .	$\frac{\mu R_v}{v}$ .		
Carbon Tetrachloride $CCl_4$ . B. Pt. $76^{\circ}7$ C. Formula Weight 153.8.										
$\times 10^6$				$\times 10^4$						
1.018	.0295	3.91	...	2.185	1.0	28.1	.0209	.49	.49	.387
1.018	.0337	7.12	348.8	"	"	114.0	.0208	.49		
1.018	.0470	6.22	348.7	"	"	74.8	.0212	.495		
1.018	.0650	8.62	350.0	"	"	161.0	.0207	.485		
1.018	.0918	12.18	350.8	"	"	300.0	.0206	.480		
1.018	.1586	21.1	352.0	"	"	[894.0]	[.0193]	[.45]		
Ether $(C_2H_5)_2O$ . B. Pt. $34^{\circ}6$ C. Formula Weight 74.08.										
1.022	.0492	6.52	...	2.96	1.0	151	.0285	.49	.49	.387
1.021	.01105	1.47	...	"	"	7.5	.0283	.49		
1.015	.0546	7.24	...	"	"	181	.0288	.495		
[1.010]	[.0548]	[7.27]	...	"	"	212	[.0277]	[.476]		
Alcohol $C_2H_5OH$ . B. Pt. $78^{\circ}3$ C. Formula Weight 46.05.										
1.010	.0716	9.53	351	4.01	1.0	506	.0389	.491	.49	.387
1.012	.0345	4.57	"	"	"	142	.039	.492		
1.015	.0607	8.05	"	"	"	394	.039	.495		
Water $H_2O$ . B. Pt. $100^{\circ}$ C. Formula Weight 18.										
Mean of the first 27 observations recorded in Table III. ....									.50	.394
$R_v$ the atmospheric pressure approx. $1.01 \times 10^6$ dynes .....										
$\mu$ , average value .033. $v_v = 6.6 \times 10^4$ .....										
II. Experiments below Atmospheric Pressure.										
Ether.										
$\times 10^6$				$\times 10^4$						
.28	.0177	2.35	287.2	2.86	1.0	67.0	.0992	.49	.49	.387
.64	.0113	1.50	300.0	2.92	"	13.3	.0454	.497		
.243	.0212	2.81	280.7	2.85	"	94.2	.116	.495		
.317	.02045	2.71	287	2.86	"	79.0	.087	.484		

We may note that the graph near the origin is parabolic at ordinary pressures because the probability of collisions is considerable and the exponential term important; at very low pressures we should expect to find it departing from this because the probability of collisions is small, so that both streams of molecules would independently affect the pressure.

Since  $\frac{\mu R_v}{v_v}$  has the same value for all liquids tested, it follows that  $\beta_i$  will also be the same for each, provided of course that  $\mu$  is always measured for temperatures and pressures which correspond to a condition of saturation. Also since  $\beta_i = .393$ , we have  $n_l = n_v(1 - \beta_i) = .606n_v$ .

Some caution is necessary in interpreting this:— $n_l$  and  $n_v$  are numbers of molecules defined on the assumption that all possess the mean molecular velocity; it indicates that, in so far as it is legitimate to replace the actual velocity distribution by this conception, a smaller number are projected in each second than those with velocities in the downward direction. It is not, however, dependent upon any assumption regarding the relative velocities of the ejected and circumambient molecules since equation (18) does not involve  $v$ .

The ratio of  $n_l$  to  $n_v$  obtained above appears reasonable, because, since there is no loss of weight of a liquid in equilibrium with its saturated vapour, we have  $n_l = n_v(1 - \beta)$ . Moreover, since the number of vapour molecules stopped cannot be greater than the number rising, we have  $\beta_i n_v \nless n_l$ . Writing  $\beta_i n_v = kn_l$ , where  $k$  varies from 0 to 1, it follows that

$$n_l = n_v - kn_l, \quad \text{or} \quad n_l/n_v = \frac{1}{1+k}.$$

This ratio may clearly have any value between  $\frac{1}{2}$  and 1. If the pressure is low we have such small densities that collisions are improbable, so  $k=0$  and  $n_l = n_v$ ; on the other hand, an increase of density is favourable for collisions and the limiting case occurs when  $k=1$  and  $n_l = \frac{1}{2}n_v$ . This happens when all projected material is stopped and would only be likely at high pressures.

Fresh experiments were undertaken to find the variation of  $n_l/n_v$  with temperature and pressure. Mr. J. Burgess kindly gave me great assistance in these investigations.

A smaller and improved form of reaction boiler or Hero's engine was constructed and suspended within a bell-jar, so

arranged that by electrical heating the liquid within could be boiled. Ether was used. After one or two attempts we found it possible to measure  $\mu$  for pressures varying from 1/4 to 1 atmosphere. Though  $\mu$  had proved to be a constant for a particular external pressure for all rates of discharge, it is now shown to be a function of the external pressure. One factor we had not taken into account in estimating the amount of interference between two opposing streams of molecules was the effect of further collisions upon those which had suffered effective collisions. Working backwards, it seems that, *for a given total number of encounters*, the chance of a collision being effective becomes less as the number of molecules in the external atmosphere increases. Possibly because subsequent encounters cause an associated pair to be resolved, or possibly because collisions are only effective when they occur between ejected molecules and newly evaporated material.

A very curious fact emerged from these experiments, which are also tabulated in Table II., namely, that, though  $\mu$  varies with the pressure employed, the value of  $\frac{\mu R_v}{v_v}$  remains

unaltered at the value we had previously found for water and other liquids at their boiling-points. It has never been found to differ very appreciably from .5. Since

$\beta_i = 1 - e^{-\frac{\mu R_v}{v}}$ , we see that  $\beta_i$  for all saturated vapours examined has the value .393, and that  $n_i/n_v = .607$ .

It is not the writer's opinion that these quantities would prove to be universal constants for all saturated vapours at all pressures. If the experiments could be conducted under sufficiently high vacua, it seems almost inevitable that the ratio of  $n_i$  to  $n_v$  will approach unity. It is desirable, but difficult, to test this.

It is certainly very surprising to find no certain variation in  $\beta_i$  when the pressure undergoes a 4 to 1 change. We should not expect to find a stationary value until  $n_i/n_v = .5$ , when  $k=1$  and all the projected material is effective in preventing the external momentum from reaching the surface. It would be interesting to find whether a statistical examination of our problem would alter our value of  $\beta_i$  from .393 to .5, which would be in accord with this point of view.

[Another possible modification of our problem also requires consideration. We have previously supposed that the  $\beta_{n_v}$



molecules which had been prevented from reaching the surface by collisions remained part of the vapour, but it is possible that some sort of binary system is formed which drifts back eventually to the surface and is to be added to the liquid. This would be in keeping with the simplification introduced into our mathematical treatment, namely, that interference operated through the complete stoppage of a few molecules, the rest maintaining their velocities unimpaired. It is interesting to note that on the assumption that binaries are formed, and that all fall back upon the liquid, equation (16 a) becomes

$$L = n_l - n_v(1 - \beta_i) - 2\beta_i n_v = n_l - n_v(1 + \beta_i) = 0,$$

and we should find  $\beta_i = .632$  and  $n_l/n_v = 1.632$ , which is not an improvement upon our earlier estimate. If, however, only the projected material which suffers interference drifts back to the liquid, we have

$$L = n_l - n_v(1 - \beta_i) - \beta_i n_v = n_l - n_v = 0.$$

This gives us in place of equation (17)

$$R = mn_v v_v(r + 1 - \beta_i),$$

whence it follows that

$$\beta_i = 1 - e^{-\frac{\mu R_v}{v_v} \frac{r+1}{r+1-\beta_i}} = 1 - e^{-.5 \frac{r+1}{r+1-\beta_i}},$$

or, if  $r=1$ ,

$$\beta_i = 1 - e^{-.5 \frac{2}{2-\beta_i}} = 1 - e^{-\frac{1}{2-\beta_i}},$$

an equation which is satisfied with considerable accuracy if  $\beta_i = .5$ , i. e. if half the material directed downwards is stopped by collisions.

We may note that our treatment of the jet experiments is not affected by this assumption, it is most improbable that any material condensed outside the vessel will re-enter it.]

It thus seems that when a liquid is in equilibrium with its saturated vapour, there is always a definite ratio between the number of molecules issuing from the surface and those directed towards it; this ratio is .607 (or possibly 1). As the boiling-point is defined as the temperature at which the vapour pressure of the liquid is equal to that of the

surrounding atmosphere, we may define the boiling-point as that temperature at which the emission is such that  $\cdot393$  (or possibly  $\cdot5$ ) of the downward momentum is prevented from reaching the liquid by the shielding action of the projected material. At a higher temperature for a given pressure, the emission increases and more molecules leave than enter the liquid, which therefore boils away; at a lower temperature the emission is less and there is condensation.

One consequence of the invariability of  $\mu R_v/v$  is that we may apply it to equation (16) when, since the discharge is small, the pressure of the vapour in the boiler will be nearly equal to the external pressure, *i. e.*  $R_v = \Pi$ , so that by substituting for  $\mu$  from the equation  $\mu \Pi/v_l = \cdot5$ , we have for the total reaction,  $R$ , upon a vessel with an aperture of area  $A$

$$R = (Dv_l)^2/2\Pi A,$$

which is a great simplification of equation (14); it holds good with considerable accuracy, but is, of course, only applicable to small rates of discharge. We are, however, now in a position to predict the reaction experienced by an orifice of any small area  $A$  in a thin plate for all discharge rates and for all liquids which behave like those we have examined. We can find  $\mu$  for the particular liquid, external pressure, and boiler temperature from the above equation; knowing  $D$  and obtaining  $v_l$  from the kinetic theory, we can at once calculate  $R$  from equation (14).

In conclusion, I wish to thank Professor J. W. Nicholson, F.R.S., for his continued interest in this research and for his encouragement to publish it. I am also indebted to Dr. Harold Jeffreys for reading it through; I do not think there is anything in it at variance with his own treatment of evaporation. I have mentioned in the text the members of my own laboratory, Messrs. H. Tunley, J. Webber, and J. S. Burgess, to whom I am very grateful for willing assistance.

University College,  
Reading.

## PART II.

## APPENDIX.

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## SECTION I.

*Experiments upon liquids evaporating into the open air.*(Vide *Part I. Argument*, page 643.)

IN the two pans of a balance similar crystallizing dishes were placed, one containing ether, and the other counterpoise weights. Each was provided with a cover-glass, that over the ether being ground to fit tightly: these were made equal in weight by pasting strips of paper on the lighter one. Balance being obtained, the two covers were removed, when the ether began to evaporate. The motion of the pointer was watched. After one minute the covers were replaced and weights added to restore the balance and to measure the amount of ether which had evaporated in the interval. In a typical experiment the rate of evaporation,  $D$ , was 0.00133 gm. per sec., the temperature was  $15^{\circ}\text{C}$ ., hence the molecular velocity,  $v$ , was approximately  $4 \times 10^4$  cms./sec., whence, assuming normal projection, the rate of loss of momentum  $= Dv = 52.5$  dynes  $= 0.054$  gm. weight. From the measured rate of loss of material 0.054 gm. would evaporate in 41 seconds: consequently, if there were no compensating action, the pan containing the liquid should have been depressed until 41 seconds from the beginning of the experiment. Observation showed that from the



start this pan rose, consequently we conclude that there is no such unbalanced pressure under the conditions of the experiment. The balance was easily sensitive to 0.001 gm. Other experiments carried out in the same manner gave similar results.

## SECTION 2.

*Note upon the mechanism whereby a boiling liquid discharges its vapour.*

At first it seemed doubtful whether the projection of molecules from a boiling liquid could be usefully examined. The bubbles originate at the base of the liquid and rise to the surface; if, as is generally held, they then burst and discharge their vapour into the space above the liquid, we could not regard the molecules as crossing the surface with anything approaching their temperature velocities, but close examination of the process does not confirm this view of the manner in which the vapour escapes. Only a very small fraction of those which reach the surface burst, the bulk of them appear to part with their vapour by a process akin to diffusion through their enclosing skins—perhaps by evaporation from their exterior surfaces accompanied by condensation of the vapour within. At any rate they gradually subside to the general level of the liquid; this is true even for the large bubbles formed by the coalescence of several small ones; it is rare for one to burst.

The phenomenon has been carefully watched in shallow and in deep vessels, and I am indebted to Mr. H. Tunley and Mr. J. Webber for checking this observation with me. The phenomenon was well shown by water boiled in the shallow lid of a square tin in which the liquid bulged high over the edges. The bubbles rose chiefly at the corners where many coalesced into large bubbles. These then floated towards the centre, growing gradually smaller until all that was left there was a froth of tiny bubbles. In a typical experiment the surface lost 0.0015 gm. per sq. cm. per second, which is a rapid rate for boiling in the open and I doubt if a single bubble burst when the surface was convex.

When the vessel is deep the bubbles rise in a stream and sometimes float on the surface, but frequently condensation takes place a little below the surface; the motion occasions a pronounced convexity which increases the surface area, and therefore the amount of evaporation. In this way a balance is arrived at between the rates at which heat is

applied and removed from the liquid. Sometimes minute droplets are thrown 4 or 5 inches above the liquid and then fall back; they do not appear to be due to condensed vapour, but to a jerking upwards of the liquid above a string of rising bubbles which collapse suddenly.

Blowing upon a bubble-strewn surface does not cause them to burst, but stops their formation below the point towards which the stream of air is directed. Bubbles of dissolved air burst when they reach the surface.

The last stage in the boiling of water in a smooth tin is the formation of a layer of water in a thin skin at the bottom, this evaporates without bubbling, even though ebullition may have been brisk when the water was deep.

### SECTION 3.

#### *Experiments upon liquids boiling in the open air.*

Since we have shown in the Argument that no delicate experiments were required in the first instance, the investigation of a possible recoil was made in much the same way as before, water being boiled on the scale-pan of a balance and the motion of the pointer observed. A flame was first applied below an empty pan and balance obtained; a small quantity of water contained in a tube was then suspended above this pan and a counterpoise placed in the other pan. It was arranged that when a wax plug in the bottom of the tube melted, the water would flow out on to the pan, where it should at once begin to boil. Any resultant reaction would be measured by a downward motion of the pan, allowance being made for the rate at which it lost weight and for convection and allied effects.

In an actual experiment 5.5 gms. of water boiled away in 3 minutes, whence, as we have seen, the quantity  $Dv$  is equal to the weight of 2.14 gms. (or 1.23 gms. if the projection is at random instead of normal); though the pointer showed small oscillations as it gradually moved in accordance with the rate at which weight was being lost, no oscillation corresponded to a pressure upon the surface as great as 0.1 gm. weight. Had there been an unbalanced reaction due to the molecular projection, the pan containing the liquid should have been depressed by a force varying from not less than the weight of 1.23 gms. at the outset to zero 40 seconds later.

In a second experiment a more sensitive test was provided, the water was contained in a small test-tube, which was

held in a vertical position by a thread ; balance was obtained with a flame below the pan and the cotton burnt ; the water then spilled on to the pan and quickly boiled away. The pointer of the balance showed a continuously increasing downward tip of the other pan. The mass of the water was 3.20 gms. and this boiled away in 20 seconds, the rate of loss being 0.16 gm. per second. The quantity  $Dv$  has the value  $1.12 \times 10^4$  dynes, which is equivalent to the weight of 11.4 gms., a quantity more than 3 times greater than the whole weight of water, a force which could not have escaped observation if it were in operation and unbalanced, and which should have persisted until all the water had boiled away. Preliminary experiments had shown that the displacement of the little tube, when the cotton was burnt, did not disturb the balance appreciably, and that convection currents provided no effect comparable with the magnitude of  $Dv$ .

#### SECTION 4.

##### *Note upon the velocities of molecules evaporating from a liquid surface.*

As we have stated in the outline of the Argument a conceivable explanation of the absence of a reaction upon an evaporating surface is that the molecules issue with zero or very small velocity. We might imagine that the molecule in the liquid state possesses just sufficient energy to carry it clear of the surface, where it remains at the absolute zero of temperature or with very small velocity until it is swept away by the vapour above and caused to share in the general motion of the rest of the molecules. The energy necessary to produce this motion would presumably be derived from the vapour, which would receive the equivalent heat from contact with, or radiation from, the hot liquid or the sides of the vessel.

This view is far from the usual interpretation of the phenomenon of boiling or evaporation, but it is difficult to quote any quite conclusive argument against it. Since distillation can take place in a vacuum it seems improbable, but provided the molecules issue with just sufficient energy to carry them clear of the surface, we could understand the liquid losing weight. The strongest argument against it lies, I think, in Knudsen's experiments (*Annalen der Physik*, xlvii. p. 697 (1915)), in which he observed the rate of evaporation of a drop of mercury *in vacuo*, and found that it agreed with a formula derived from the kinetic theory on

the assumption that evaporation was the reverse process of condensation, and, therefore, that the molecules escaped with the velocity of the vapour molecules at the same temperature. The formula had previously been obtained by Langmuir.

The writer thought it possible to test this matter further, so devised some experiments whereby it was sought to investigate the character of the reaction upon a surface evaporating in a vacuum. If evaporation from such a surface were suddenly permitted, it seemed that the reaction should attain its maximum value instantaneously if the molecules escaped with their molecular speeds, but that if they merely "dropped off," it would take an appreciable time for the molecules to acquire velocities appropriate to the temperature by radiation or contact with the sides of the vessel, or, in other words, that their vapour pressure would take an appreciable time to develop.

The apparatus described in Section 5 was employed, and it is sufficient here to say that (1) a reaction of a powerful nature was observed, and (2) that it appeared to be of a sudden explosive character, as though the molecules were projected with initial speeds of great magnitude.

The accumulated evidence is thus, in the writer's opinion, in favour of molecular projections with high velocities, though whether these are appropriate to the temperature as given by the kinetic theory was not determined. As far as our evidence goes it favours Knudsen's view. We may, however, note that the precise value of the velocity does not enter into our final equations dealing with the phenomena of evaporation.

## SECTION 5.

### *Experiments upon a liquid evaporating in vacuo.*

The experiments devised to observe if there is any recoil *in vacuo*, and referred to in Section 4 and on p. 649 *et seqq.* of the Argument, were carried out as shown in fig. 9. Fig. 10 shows a larger scale elevation of the tap A. The experiment consisted in filling the hole drilled through this tap half with mercury and half with water when the tube was vertical, and then turning it round until the tube was horizontal, when both ends were simultaneously thrown into communication with the tubes B and C, which were connected to the equally exhausted reservoirs D. If the molecules possess sensible velocities of escape they will occasion a reaction upon the liquid surface, which will result



in the expulsion of the mercury from the tap into the tube B. Preliminary experiments showed the importance of eliminating any layer of air in the interface between the mercury and the water, because if it is present, the mercury will go off in one direction and the water in the other. To get rid of the air, the water and the mercury were first boiled separately and then brought together in a reservoir below F.

Fig. 9.

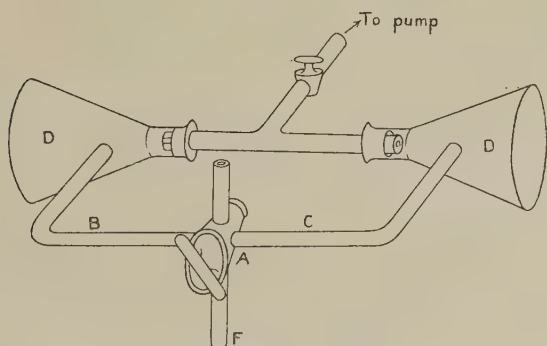
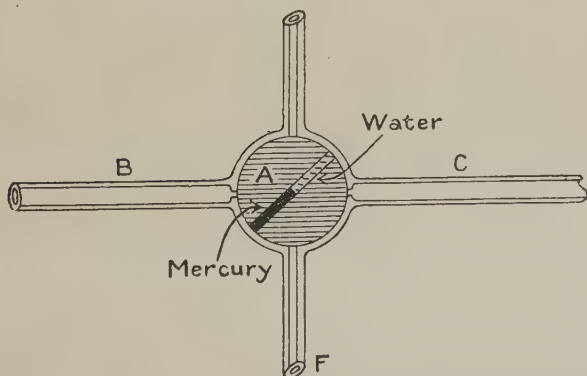


Fig. 10.



This was boiled so that steam filled the tube F and the tap, and finally, when the tube was thus as free from air as possible, the water and mercury were sucked up into the tap until it was half full of both; the tap was then turned through  $45^\circ$ , while DD were exhausted. On turning tap A quickly until the tube through it was horizontal, the behaviour of the mercury was observed.

With one exception, possibly due to a leak in the tap, a large number of experiments agreed in showing a recoil of

the mercury in the direction to be expected if the molecules were projected from the water with high speeds. It had been hoped to obtain an accurate measure of the pressure at which the recoil first became apparent, but an improved form of the apparatus ordered twelve months ago has not yet been delivered, so this extension of the investigation has been postponed.

The feature of the experiment was the sudden, explosive character of the reaction, the mercury being blown violently into the tube behind it, sometimes being shattered into a number of small drops.

In devising the experiment it had been assumed, apparently with justification, that the rate of evaporation of mercury at the temperature of the experiment would be negligible compared with that of water.

In Section 10 is quoted an experiment carried out by Osborne Reynolds which, as he interprets it, seems to indicate that at pressures below about 2.5 mm. there is a reaction when water evaporates.

## SECTION 6.

### *The discharge of molecules from an aperture in a boiler.*

The apparatus used to test the reaction due to the projection of steam from an aperture was of such a form that the molecules, which were originally emitted vertically from the horizontal surface of a liquid, escaped horizontally from an aperture whose plane was vertical. Thus all molecules which escaped had either themselves struck the side opposite the aperture, or had collided with others which were themselves driven back upon it.

As far as the reaction due to the escaping molecules is concerned, we may thus regard a small area of the opposite wall as the emitting surface, and, since we are only concerned with the component of the molecular velocity normal to the aperture, we may regard this area as the geometric shadow of the aperture itself.

Any interference which takes place between the issuing material and that incident from outside will occur somewhere along the jet based upon this small area, and may happen either within, or, more probably, outside the vessel. Along this jet it should be possible to find some section which serves as the boundary between regions in which interference does and does not take place, it marks the limit to which the external molecules can penetrate and still be liable to interference; all external molecules which succeed in reaching

it will affect the reaction experienced by the vessel in the ordinary way. This may be the wall itself or a plane near the aperture, or even some section beyond the aperture. We shall designate this section the "effective aperture" of the vessel. Our object in defining it is to indicate that it corresponds to the surface of an evaporating liquid so far as our particular problem is concerned.

We may note that if the effective aperture is within the vessel, it is probable that external vapour molecules will be condensed within it, whereas, if it is outside, such molecules will not necessarily, or even probably, be captured, though their momentum cannot fail to be conveyed by elastic collisions to the opposite wall.

The problem of the discharge of steam from such a vessel is thus very similar to the projection of molecules from a liquid in the process of evaporation. In the former, the molecules which pass outwards through the effective aperture will have occasioned a reaction upon the opposite wall given by their mass multiplied by the normal component of their molecular velocity, and the total force upon the vessel is compounded of this reaction, together with that due to the incidence of the external atmosphere after it has suffered encounters with the issuing stream.

There may, however, be differences in the distributions of the velocities of the discharged molecules about the normals to the evaporating surface and the aperture respectively; at present we make no assumptions concerning them. We may note, however, that from an investigation upon quite different lines, Knudsen (*Annalen der Physik*, xlvii. p. 697 (1915)) finds that the evaporation of molecules from a liquid surface *in vacuo* is equivalent to the efflux of gas molecules from an aperture of the same dimensions. In the writer's opinion the resemblance extends to discharges into the atmosphere.

## SECTION 7.

### *Experiments upon the reaction consequent upon the emission of steam from an orifice in a boiler.*

The experiments were performed with a piece of apparatus of the general form of Hero's engine:—a cylindrical brass vessel 3.1 cms. internal diameter was provided with four radial arms (7.5 cms. to centre) as shown in fig. 1. It contained water and was heated by a flame below it.

Steam issued tangentially and provided a torque which was measured by noting the twist given to a torsion fibre, a

circular scale being attached to the apparatus and read against a fixed pointer. There was at first a good deal of irregularity in the rate of boiling, so it was necessary to take the mean of a large number of swings on both sides, but with experience, the amplitudes of the oscillations were greatly reduced. The apparatus was weighed cold before and after each run, so that the rate of loss of steam could be determined.

A thermometer with its bulb in the steam gave the temperature, and in some experiments a U-tube gauge was used to give the pressure, but when the excess within was small this could not be determined with much accuracy. The apertures were holes drilled in thin sheet brass, and between experiments were varied in number and size.

Convection currents were found to produce no appreciable effect. After the flame was applied, but before the steam issued, the pointer remained at zero. It also returned to zero immediately the flame was removed and steam ceased to escape, though the apparatus was still very hot. While steam was actually issuing no variation of the reaction could be produced by screening the rear parts of the arms from which the steam issued; also, as we have stated, the introduction of an obstacle within 2.5 mm. of the orifice and within the jet had no effect.

The first experiments related to the efflux of steam from the four apertures fully open, when the sum of their areas was equal to that of the cross-section of the central tube in which the water was boiling. The temperature of the steam in the orifice was sensibly the same as that immediately above the liquid. The first three experiments quoted in Table I. show that the observed reaction was either zero or differed from zero by a negligible quantity. In order to show that the interactions with the atmosphere are of great importance, we may contrast this with the rate at which the vessel was losing momentum from the escape of vapour molecules on the supposition that this is the only factor. We have already indicated that the velocity of rebound from the wall opposite the aperture previous to escape is of the order of magnitude of that with which the molecule leaves the liquid: consequently the rate of change of momentum on the assumption that the jets are parallel, which is not quite the case, may be calculated from the quantity  $Dv$ , where  $D$  is the number of gms. discharged in one second, and  $v$  is of the order of  $7 \times 10^4$  cms. per second for steam at  $100^\circ \text{C}$ . Thus in the three experiments the



deflexions which would have been occasioned by these forces, if there were no compensating action, would have amounted to  $6300^\circ$ ,  $32000^\circ$ , and  $9510^\circ$  respectively. Since in the actual experiments the deflexion was sensibly zero, we must conclude that in this case, as in the evaporation experiments, the outgoing molecules encountered the molecules of the external medium, which were advancing towards the aperture, and prevented them from reaching the vessel and communicating their momenta to it. These experiments may be said to confirm our view of the interference of the two streams of momentum.

The remaining observations recorded in Table I. show that as the number of apertures was reduced from four to one a reaction upon the vessel was observed, but that it was always very small. It is, however, clear that we must reduce our aperture if we wish to increase the reaction for a given rate of discharge.

Table III. records a number of experiments made by Mr. Tunley and the writer using small orifices varying from  $\cdot 1$  to  $\cdot 6$  cm. in diameter, and also deals with a special set by Mr. Burgess and the writer made upon an orifice only  $\cdot 05$  cm. in diameter.

Data for higher rates of discharge are available from a series of experiments conducted by Rosenhain in 1900, in an investigation upon nozzles used in steam turbines; his experiments were carried out on a large scale, with an aperture of  $\cdot 475$  cm. in a thin plate, the steam being produced in an engine-boiler under pressures up to 200 lb. per sq. inch. The steam escaped in a horizontal jet from a chamber suspended by a flexible tube, the reaction being measured by means of a weight attached to a cord passing over a pulley, which gave the chamber a horizontal thrust.

The rate at which steam was discharged was ascertained by condensing it in a vessel surrounding the jet. Though the object of the experiments was entirely different from that of those under discussion, it is possible to derive the necessary data from cross reference to Rosenhain's diagrams. This is given in the final section of Table III. Attention is for the present drawn to the observations and quantities deduced directly from them; the calculated values will be discussed later.

Table IV. records observations made by Rosenhain when nozzles of various forms were substituted for the orifice in a thin plate.

TABLE III.

Observers : J. S. Burgess, H. Tunley, G. Duffield.

No. of Apertures.	Diam. of Aperture.	Dis-charge per second. D.	Dis-charge per sec. per unit area. D/A.	Temp. of steam. T <sub>Abs.</sub>	Molecular velocity. $v_L$ .	Ratio $v_L/v_{10c}$ . $r$ .	Reaction observed. dynes. $R_o$ .	Reaction calculated. dynes. $R_c$ .
1	.53 mm.	.0133	6.03	374.5	$6.60 \times 10^4$	1.0	167	165
		.0180	8.2	376.0	6.61	1.00	267	285
		.0256	11.6	379.5	6.66	1.01	519	522
		.0283	12.8	379.6	6.67	1.01	590	632
		.0308	13.95	382.0	6.72	1.02	715	735
1	1 mm.	.0076	.96	373	6.60	1.0	22	12
		.0094	1.19	"	"	"	18	20
		.0149	1.89	"	"	"	51	56
		.0178	2.26	"	"	"	60	78
		.0260	3.30	"	"	"	125	155
		.0239	3.03	"	"	"	195	140
		.0313	3.97	"	"	"	212	230
		.0088	.28	"	"	"	5.6	6
1	2 mm.	.0099	.31	"	"	"	6.2	9
		.0180	.57	"	"	"	22	27
		.0255	.82	"	"	"	45	46
		.0334	1.06	"	"	"	75	76
		.0380	1.21	"	"	"	109	100
		.0420	1.34	"	"	"	133	120
		.0443	1.41	"	"	"	133	138
		.0500	1.59	"	"	"	172	180
1	3 mm.	.0124	.18	"	"	"	6.2	6
		.0220	.31	"	"	"	17	16
		.0237	.34	"	"	"	19	18
		.0310	.44	"	"	"	36	33
		.0330	.47	"	"	"	42	38
		.0559	.79	"	"	"	122	106
1	4 mm.	.0326	.26	"	"	"	20	
1	5 "	.0351	.18	"	"	"	16	
1	6 "	.0358	.13	"	"	"	15	
2	1 "	.0206	1.31	"	"	"	58	
2	2 "	.0470	.75	"	"	"	89	
3	2 "	.0359	.57	"	"	"	42	
		.0199	.21	"	"	"	10	
		.0390	.41	"	"	"	49	
4	2 mm.	.0344	.27	"	"	"	21	
		.0370	.29	"	"	"	23	
		.0333	.23	"	"	"	20	
		.0275	.19	"	"	"	15	
		.0271	.13	"	"	"	14	
		.0245	.12	"	"	"	12	
		.0209	.07	"	"	"	6	
		.0417	...	"	"	"	19	
		.0480	...	"	"	"	18	
		.0320	...	"	"	"	5	

TABLE III. (*cont.*).  
High Pressure Discharge (Rosenhain).  
Diameter of Aperture 4.75 mm.

Boiler Pressure lb. per sq. in.	D.	D/A.	T <sub>calc.</sub>	$v_l$	$r$	R <sub>obs.</sub>	R <sub>calc.</sub>
20	6.1	34.5	399	$7.46 \times 10^4$	1.04	$.29 \times 10^6$	$.29 \times 10^6$
40	9.7	54.7	415	7.60	1.06	.58	.56
50	11.3	63.9	421	7.66	1.07	.73	.69
100	20.3	114.5	443	7.86	1.10	1.44	1.41
150	28.0	158.0	459	8.0	1.12	2.08	2.05
200	(35.4)	(200.0)	471	8.1	1.13	(2.82)	2.67

Note upon Table III.:—

(1) *Low pressure experiments.*—Though the application of the quantities recorded in the Table is reserved for later discussion, we may state here the manner in which they were evaluated.

D was found by weighing, as described in the text. The diameter of the 0.53 mm. orifice was obtained with the help of a travelling microscope; the other orifices were less carefully measured. The temperatures were those given by the inserted thermometers after correcting for stem-exposure, etc. Only in the case of the 0.53 mm. aperture did this differ appreciably from 373° Abs.

The molecular velocity was calculated from the formula

$$\bar{V} = 18.4 \times 10^4 \sqrt{\frac{2}{18} \frac{T}{273}},$$

and the value of  $v_l$  taken as .921  $\bar{V}$ , so that in the low-pressure experiments it is the mean velocity of the molecule.

In calculating  $r$ , the ratio of the velocity of the escaping molecules to that of those outside, the external medium was taken to be steam at 100° C.; it is difficult to be sure of this, but the value of  $r$  is not a very important quantity in our subsequent equations. Upon no reasonable assumption could it differ greatly from unity.

(2) *High pressure experiments.*—Unfortunately Rosenhain's published data are incomplete (Proc. Inst. C. E. vol. cxl. pt. ii., 1899–1900), and our measurements have been taken from his diagrams. The boiler temperatures are not stated, so we have estimated them on the assumption that the steam was saturated vapour at the given boiler pressure by using steam tables. But it is probable that the temperature was a little higher than the value thus obtained, because, since steam was escaping rapidly, the measured pressure was probably less than that of stagnant saturated vapour corresponding to the actual temperature. We have calculated  $\bar{V}$  as before, but  $v_l$  has been given the R.M.S. value, instead of the mean value, as was done when the temperature was actually observed.

TABLE IV.  
The Reaction upon Nozzles (Rosenhain).

Nozzle.	Length. cms.	Maximum Aperture. sq. cms.	Taper.	Boiler Pressure in lb. per sq. in.									
				200		140		80		40		20	
				R.	D/A.	R.	D/A.	R.	D/A.	R.	D/A.	R.	D/A.
I ...	0.177	none	$\times 10^6$ 2.82 (200)			$\times 10^6$ 2.0	151	$\times 10^6$ 1.15	94.5	$\times 10^6$ 0.58	54.7	$\times 10^6$ .29	34.5
II ...	5.3	.417	1 in 20	3.44	94.8	2.35	68.6	1.22	42.5	0.49	12.0	.22	5.45
III ...	5.5	.686	1 in 12	3.15	53.5	2.07	43.0	1.11	23.8	0.33	13.9	.16	8.6
III A.	2.0	.329	1 in 12	3.30	109	2.24	78.5	1.18	48.3	0.49	27.6	.20	17.9
IV D.	1.7	.212	1 in 30	2.88	171	2.05	125	1.16	77	0.53	4.5	.22	(21.4)

The nozzles were of the expanding type. The inlet edges of  
III, III A, and IV D were slightly rounded.

### SECTION 8.

#### *Theoretical considerations relating to the phenomenon of evaporation.*

We proceed next to consider in more detail the conditions which determine the pressure upon an evaporating surface.

The pressure is made up of two parts: (*a*) that due to the impact of molecules of vapour from above—we shall assume in the first instance that the external medium close to the surface consists entirely of the vapour of the liquid; and (*b*) that due to the rate of loss of momentum from the evaporating surface.

Let  $n_v$  be the number of vapour molecules which, if there were no loss by evaporation from the liquid, would strike a unit plane area within the vapour space close to the liquid in one second, let  $v_v$  be the mean velocity component of these molecules normal to the surface, and  $m$  the mass of each; then the pressure upon unit area  $\Pi$  due to these is given by the momentum delivered in each second, namely

$$\Pi = 2mn_v v_v. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If now evaporation takes place, it is clear that the momentum of which we have been speaking will encounter an opposing stream of rising molecules, and that collisions



will tend to reduce the number penetrating to the surface. Some of the molecules will have their velocities reduced to zero, so that they do not reach the surface at all.

Let us regard the loss of momentum of the downward stream as equivalent to the complete stopping of a fraction  $\beta$  of the original  $n_v$  molecules.

Thus the number of molecules actually reaching the liquid is  $(1-\beta)n_v$ ; of these let the fraction  $\alpha$  be reflected, so that the number penetrating the surface is  $n_v(1-\beta)(1-\alpha)$ . Since it is possible that the surface discriminates between fast and slow moving molecules, let us write  $v_v'$  for the normal component of the velocity of those which penetrate it, and  $v_v''$  for those which do not.

Let  $n_l$  molecules leave unit area of the surface per second on account of the evaporation process, and let  $v_l$  be their mean velocity component normal to the surface. Then the number of molecules,  $L$ , permanently lost by each unit of area in one second is the difference between  $n_l$  and those which penetrate the surface, or

$$L = n_l - n_v(1-\beta)(1-\alpha). \quad (2)$$

When a liquid is evaporating freely into the atmosphere, for example from the pan of a balance, the conditions close to the surface are as represented in the accompanying

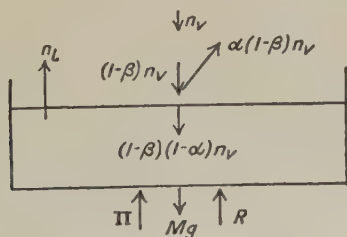


figure. Here  $R$  is the applied force necessary to secure equilibrium, and  $\Pi$  the pressure of the atmosphere acting upwards upon the base of the pan. The arrow marking  $n_v$  is supposed to be sufficiently removed to be free from the effect of the rising molecules.

Let  $Mg$  be the weight of the vessel and its contents at any instant, then, since each escaping molecule contributes to the resultant downward pressure a quantity  $mv_l$ , each molecule reflected from the surface an amount  $2mv_v''$ , and

each molecule which penetrates the surface an amount  $mv_v'$ , we have

$$R + \Pi - Mg = mn_l v_l + 2\alpha(1-\beta)mn_v v_v'' \\ + (1-\beta)(1-\alpha)mn_v v_v'. \quad . \quad . \quad (3)$$

The natural limits between which  $\alpha$  and  $\beta$  may lie are 0 and 1, but the available evidence for a clean surface indicates that  $\alpha=0$  (Bennewitz, *Annalen der Physik*, lix. p. 193 (1919); Knudsen, *ibid.* xlvii. p. 697 (1915)). Accepting this value, it follows that  $v_v = v_v'$ , since the whole of the advancing stream of momentum  $m(1-\beta)n_v v_v$  penetrates the surface. Substituting these values for  $\alpha$  and  $v_v'$ , it follows that

$$R + \Pi - Mg = mn_l v_l + (1-\beta)mn_v v_v, \quad . \quad . \quad (3a)$$

or, since we may suppose the instantaneous value of  $Mg$  to be balanced by a weight in the other pan, we have, if we substitute for  $\Pi$  from (1)

$$R = mn_l v_l - (1+\beta)mn_v v_v. \quad . \quad . \quad . \quad (4)$$

Also from (2)

$$L = n_l - (1-\beta)n_v. \quad . \quad . \quad . \quad (5)$$

It is not possible to observe  $n_l$  directly, but, using its value as given by (5) in (4), we obtain

$$R = mLv_l + mn_v(1-\beta)v_l + (1+\beta)mn_v v_v. \quad . \quad . \quad (6)$$

We may also write

$$r = v_l/v_v, \quad . \quad . \quad . \quad . \quad (7)$$

where  $r$  is the ratio of the velocity of the molecules escaping from the liquid to that of those in the vapour space.

We therefore have for the reaction in excess of the atmospheric pressure at any moment

$$R = mLv_l - mn_v v_v \{ \beta(r+1) - (r-1) \}, \quad . \quad . \quad (8)$$

$$\text{or} \quad R = mLv_l - \frac{\Pi(r+1)}{2} \left\{ \beta - \frac{r-1}{r+1} \right\} \quad . \quad . \quad (9)$$

If we deal with area  $A$  the mass of the discharge per second,  $D$ , may be written

$$D = mL A, \quad . \quad . \quad . \quad . \quad (10)$$

whence the total reaction is given by

$$R = Dv_l + \frac{\Pi A}{2}(r-1) - \frac{\Pi A}{2}(r+1)\beta. \quad . \quad . \quad (11)$$

*Note.*—For the particular case in which the liquid and vapour are at the same temperature we have  $v_l = v_v$  and  $r = 1$ . Writing  $r = 1$  in (11) we have

$$R = Dv_l - \beta \Pi A, \quad . \quad . \quad . \quad (12)$$

or from equation (8)

$$R = mv_v A \{L - 2\beta n_v\}. \quad . \quad . \quad . \quad (13)$$

If, as on page 647 of the Argument, we suppose  $\beta = 1$  when  $r = 1$  (a condition unattainable in practice but nevertheless of interest in what follows), we should have  $R = 0$  until  $L$  exceeded  $2n_v$ , i. e. until twice as many molecules were emitted as would have been incident upon the surface if there had been no interference. This is equivalent to writing  $Dv_l = \Pi A$ . For  $\Pi = 10^6$  dynes, unit area, and  $v_l = 6 \cdot 6 \times 10^4$  cms. per second for steam at  $100^\circ \text{C}$ ., the maximum value of  $D$  attainable theoretically without a reaction is given by  $D = 10^6 / 3 \cdot 6 \times 10^4 = 15$  gms. approximately.

## SECTION 9.

### *Theoretical investigation relating to the efflux of vapour molecules from a boiler under steady conditions.*

The equations arrived at from the consideration of the conditions under which a liquid evaporates (Section 8) are without modification applicable to the discharge of vapour from an orifice. For steam escaping from an aperture in the vertical side of a boiler, as in the apparatus shown in fig. 1, we see that the case is simpler, because there is no need to take account of the weight of the apparatus, also as there can be no reflexion at the aperture of molecules advancing towards it from outside, we have  $\alpha = 0$  as before.  $R$  is now measured by the torsion of the wire, and we have equation (11) holding good, namely

$$R = Dv_l + \frac{\Pi A}{2}(r-1) - \frac{\Pi A}{2}(r+1)\beta, \quad . \quad . \quad (11)$$

where  $v_l$  now applies to the molecules leaving the "effective aperture."

## SECTION 10.

*Previous experiments bearing upon an evaporative reaction.*

Osborne Reynolds has described an experiment in which two pith balls were suspended at the ends of a light beam by a torsion fibre within a vessel containing water-vapour under reduced pressures. It was found that at pressures below about 2.5 mm., a flame held outside the vessel appeared to repel the nearest pith ball and a lump of ice to attract it. Reynolds \* explains the phenomenon in these terms:—

“According to the kinetic theory, the molecules which constitute the gas are in rapid motion, and the pressure which the gas exerts against the bounding surface is due to the successive impulses of these molecules, whose course directs them against the surface, from which they rebound with unimpaired velocity. According to this theory, therefore, whenever a molecule of liquid leaves the surface henceforth to become a molecule of gas, it must leave it with a velocity equal to that with which the other particles of gas rebound, that is to say, instead of being just detached and quietly passing off into the gas, it must shoot off with a velocity greater than that of a cannon ball. Whatever may be the nature of the forces which give it the velocity, and which consume the latent heat in doing so, it is certain, from the principle of the conservation of momentum, that they must react on the surface with a force equal to that exerted on the molecule, just as in a gun the pressure of the powder on the breech is the same as on the shot.

“The impulse on the surface from each molecule which is driven off by evaporation must therefore be equal to that caused by the rebound of one of the reflected molecules, supposing all the molecules to be of the same size; that is to say, since the force of rebound will be equal to that of stopping, the impulse from a particle driven off by evaporation will be half the impulse received from the stopping and reflection of a particle of gas at the surface. Thus the effect of evaporation will be to increase the number of impulses on the surface; and although each of the new impulses will only be half as effective as the ordinary ones, they will add to the pressure.”

This is a delightfully clear picture of the molecular motions near an evaporating surface, but it does not take into account the interference between the opposing streams of molecules, which we have shown to be a vital part of the phenomenon of molecular discharges.

\* O. Reynolds, Roy. Soc. Proc. A. p. 401 (1874).



LXVI. *The Whirling of an Overhung Eccentrically Loaded Shaft.* By S. LEES, M.A., St. John's College, Cambridge\*.

*Introduction.*

§ 1. **I**N the Phil. Mag. vol. xxxvii. p. 515 (1919), the author contributed a paper under the above title, dealing with the problem of a whirling overhung shaft of negligible weight provided at its unsupported end with a disk flywheel which is not perfectly balanced. The formula for the whirling speed given in equation (18) of the paper agrees with the result obtained by Chree †, whilst the steady motion given by equation (17) of the paper is new, so far as the author knows. Unfortunately, some serious errors arise in the paper, and the periods obtained are wrong. In addition, the author's results indicated that when the whirling speed was exceeded, the shaft became unstable. This is contrary to the results of observation. The author is indebted to Prof. W. McF. Orr, F.R.S., not only for pointing out these errors, but for giving him valuable criticisms as to the method of attack adopted in the paper referred to. These criticisms have led to the following paper, in which the whole of the questions involved are redealt with, and in addition, some new points of view are given.

In connexion with the previous paper (*loc. cit.*) by the author on this subject, it was at first objected to by Prof. Orr that the use of Lagrange's equations in obtaining the results there found was not legitimate. It would now appear that the chief objection to the author's development lay not in such use of the Lagrangian method, which is here quite legitimate, but in the choice of coordinates. These were not sufficiently general to define the dynamical system under all circumstances. Thus (and the same error was made by Chree) the author assumed that the shaft could be treated as if its geometrical axis were constrained to rotate round the axis of bearings. The present paper endeavours to show that whilst this is no doubt true for steady motion, it is not true in general. Thus the present paper verifies Chree's result for the whirling speed and the author's result for the steady bending for any given speed, but for the periods gives quite different results from those obtained previously by the author. It would appear that the problem really discussed in this previous paper by the author was that of the whirling of a shaft compelled to remain in one diametral plane, which plane is itself compelled to rotate with the shaft at the constrained end.

\* Communicated by the Author.

† Phil. Mag., May 1904.

*Neglect of Torsional Effects.*

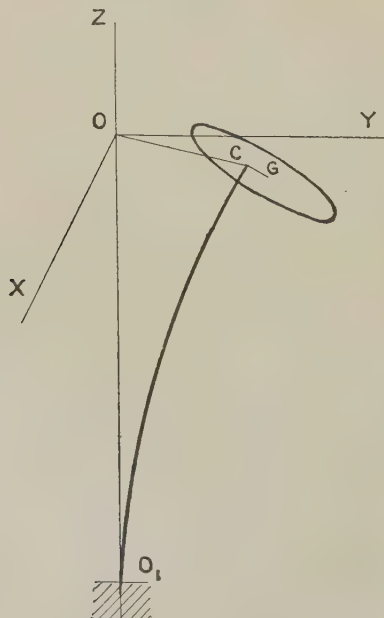
§ 2. In the paper referred to and in what follows the assumption is made that the whirling load can be taken as a perfectly true disk flywheel mounted slightly eccentrically on the shaft. It is also assumed that torsional effects are negligible. This last proviso requires a slight discussion.

It is clear that for negligible twisting of the shaft, we must have (1) the eccentricity of the load, *i. e.*, the displacement of the centre of gravity of the flywheel from the axis of the shaft when at rest, small compared with the radius of gyration  $K$  of the flywheel about the axis of the shaft; (2) the time of a torsional oscillation small compared with the time of an oscillation involving pure bending only. A discussion of these two times of oscillation shows that the length  $L$  of the shaft from the bearing of the flywheel must be large compared with  $K$ . Throughout the present paper, this will be assumed, and torsional effects will be neglected.

*Coordinates specifying the Configuration.*

§ 3. To specify the position of the shaft, we shall employ four coordinates. Referring to fig. 1, we shall take it that

Fig. 1.



the deflexions of the shaft from the straight original condition are small, and that the movement of the free end of the

axis parallel to the direction of the axis at the constrained end is absolutely negligible. The free end C of the axis may therefore be considered to move in a plane perpendicular to the axis of bearings. The axes of reference will accordingly be taken such that the constrained axis coincides with  $Oz$ , whilst  $Ox$  and  $Oy$  lie in the plane just referred to. The coordinates of C will thus be  $(x, y, 0)$ . In addition, we shall require the direction-cosines of the tangent to the axis at its free end. These will be taken as  $(\lambda, \mu, 1)$ . The four coordinates specifying the position of the shaft are thus  $x, y, \lambda, \mu$ .

We have yet to define the position of the centre of gravity G of the flywheel. The distance CG (the eccentricity of the flywheel) will be denoted by  $b$ . It may easily be shown that, provided torsional effects may be neglected, the angular motion of the line CG is the same as that of the shaft at its constrained end. This result, which is purely kinematical, arises as follows:—

Assume that the shaft is of radius  $b$ , so that G lies on its outer surface (see fig. 2). By taking the generator on the

Fig. 2.



surface of the shaft which passes through G, we obtain, corresponding to G, a point Q on the cross-section of the

shaft at the constrained end  $O_1$ . If we take on this cross-section a point  $R$ , so that  $O_1R$  is parallel to the plane  $zOx$ , we can find the corresponding point  $S$  on the cross-section at the free end lying on the generator through  $R$ . If  $O_1$  represent the centre of the shaft at the constrained end, we must have in all cases angle  $RO_1Q = \text{angle } SCG$ . If the shaft be compelled to turn uniformly with angular velocity  $\omega$  at the constrained end (by means of a heavy flywheel at the constrained end), then the angle  $SCG$  must increase at the constant rate  $\omega$ . Now the relationship between  $S$  and  $R$  is one purely depending on the *shape* of the bent shaft. If, then, we consider the purely geometrical relationship involved in the most general type of small bending deformation\* of the shaft (*i. e.*, not necessarily assuming that the axis of the shaft lies in one plane), we see that to the degree of approximation here taken, the projection (on the plane  $RO_1Q$ ) of  $SC$  is always parallel to  $RO_1$ , and, in consequence,  $CG$  rotates in the plane  $xOy$  with uniform angular velocity  $\omega$ . This result is clearly independent of the assumption that the shaft is of radius  $b$ .

We are thus entitled to take as the coordinates of  $G$  the values

$$x_G = x + b \cos(\omega t + \alpha), \quad y_G = y + b \sin(\omega t + \alpha), \quad z_G = 0. \quad (1)$$

Here  $\alpha$  represents the angle which  $CG$  makes with  $Ox$  when  $t=0$ .

### *Equations of Motion.*

§ 4. In order to get the equations of motion involving  $\lambda$  and  $\mu$ , we make use of the following theorem: If a rigid body have moments of inertia  $A, B, C$  (with  $B=A$ ) about mutually perpendicular principal axes  $OA, OB, OC$  respectively, where  $O$  is fixed relatively to the body and is either fixed in space or, if in motion, coincides with the centre of gravity, and if the angular velocities  $\omega_1, \omega_2$  about the principal axes  $OA, OB$  respectively be always small compared with  $\omega_3$ , the angular velocity about  $OC$ , then correct to the first order of small quantities:—

Moment of momentum about

$$Ox = C\lambda\omega_3 - A\dot{\mu}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and about

$$Oy = C\mu\omega_3 + A\dot{\lambda}. \quad . \quad . \quad . \quad . \quad (3)$$

\* A deformation due to pure bending might be defined as one in which the projection of any straight line in the cross-section of the shaft on a plane normal to the fixed end of the axis remains parallel to its original direction.



Here the axes OX, OY, and OZ (mutually perpendicular) are fixed, and OZ is such that OC nearly coincides with it; also  $\lambda$ ,  $\mu$ ,  $\nu$  are the direction-cosines of OC referred to O*x*, O*y*, O*z* respectively, so that  $\lambda$  and  $\mu$  are small, and  $\nu$  may be taken as unity.

To prove this theorem, we note that

$$\omega_x = \omega_1 \cos AOX + \omega_2 \cos BOX + \omega_3 \cos COX,$$

and hence moment of momentum about

$$\begin{aligned} OX &= A(\omega_1 \cos AOX + \omega_2 \cos BOX) + C\omega_3 \cos COX \\ &= A(\omega_x - \lambda\omega_3) + C\lambda\omega_3. \quad \dots \dots \dots (4) \end{aligned}$$

Now the point ( $\lambda$ ,  $\mu$ ,  $\nu$ ) lies on OC at unit distance from O; hence such point has velocities

$$\dot{\lambda} = \nu\omega_y - \mu\omega_z, \quad \dot{\mu} = \lambda\omega_z - \nu\omega_x, \quad \dot{\nu} = \mu\omega_x - \lambda\omega_y,$$

and it follows that

$$\begin{aligned} \mu\dot{\nu} - \nu\dot{\mu} &= \omega_x(\mu^2 + \nu^2) - \lambda\mu\omega_y - \nu\lambda\omega_z \\ &= \omega_x - \lambda(\lambda\omega_x + \mu\omega_y + \nu\omega_z) = \omega_x - \lambda\omega_3. \end{aligned}$$

Under our assumptions  $\nu$  is constant at the value unity; hence (4) may be written  $-A\dot{\mu} + C\lambda\omega_3$ , which agrees with (2). The expression (3) is similarly obtained.

To get the forces acting on the shaft due to independent bendings in two perpendicular planes, twisting being neglected, we take for the potential energy

$$V = 2EA_1k^2\{3(x^2 + y^2) - 3L(\lambda x + \mu y) + L^2(\lambda^2 + \mu^2)\}/L^3. \quad (5)$$

This may be regarded as a generalization of the result given by Rayleigh ('Sound,' vol. i. § 183). The notation is such that E represents Young's Modulus for the material of the shaft,  $A_1$  represents the area of cross-section which has a radius of gyration  $k$  about any diameter, whilst  $L$  is the length of the shaft from the bearing to the free end. For convenience, we shall write  $B_1 = 2EA_1k^2/L^3$ .

From (5), the restoring force parallel to O*x* is

$$\frac{\partial V}{\partial x} = 3B_1(2x - L\lambda), \quad \dots \dots \dots (6)$$

whilst the restoring couple parallel to the plane  $zOx$  is

$$\frac{\partial V}{\partial \lambda} = B_1(-3Lx + 2L^2\lambda). \quad \dots \dots \dots (7)$$

Similar results hold for the restoring force parallel to O*y*, and for the restoring couple parallel to the plane  $zOy$ .

The equations of motion of the system of shaft and eccentrically loaded flywheel are now obtained in the form

$$\frac{d^2}{dt^2} [x + b \cos (\omega t + \alpha)] + 3B_1(2x - L\lambda)/M = 0, \quad (8)$$

$$\frac{d^2}{dt^2} [y + b \sin (\omega t + \alpha)] + 3B_1(2y - L\mu)/M = 0, \quad (9)$$

$$\frac{d}{dt} (C\lambda\omega - A\dot{\mu}) - B_1(-3Ly + 2L^2\mu) = 0, \quad (10)$$

$$\frac{d}{dt} (C\mu\omega + A\dot{\lambda}) + B_1(-3Lx + 2L^2\lambda) = 0. \quad (11)$$

In the equations (8) and (9)  $M$  stands for the mass of the eccentrically loaded flywheel; whilst in (10) and (11) the moments of the restoring forces round the axes through  $G$  are neglected (being of the second order).

#### *Solution of the Equations of Motion.*

§ 5. The effect of the terms in  $\frac{\cos}{\sin} (\omega t + \alpha)$  in the above equations is clearly that of a disturbing force. To find the "free" oscillations, we have to solve

$$(D^2 + 6B_1/M)x - 3B_1L\lambda/M = 0, \quad (12)$$

$$(D^2 + 6B_1/M)y - 3B_1L\mu/M = 0, \quad (13)$$

$$-3B_1Lx + (AD^2 + 2B_1L^2)\lambda + C\omega D\mu = 0, \quad (14)$$

$$-3B_1Ly + (AD^2 + 2B_1L^2)\mu - C\omega D\lambda = 0. \quad (15)$$

In these equations  $D$  denotes  $d/dt$ . To solve them, we write

$$x = r \cos (pt + \gamma), \quad y = r \sin (pt + \gamma), \quad \lambda = \theta \cos (pt + \gamma), \\ \mu = \theta \sin (pt + \gamma). \quad (16)$$

Here  $r$  and  $\theta$  are constants which are connected, however, so that only one of them is arbitrary. To determine  $r/\theta$  and  $p$ , we have

$$(-p^2 + 6B_1/M)r - 3B_1L\theta/M = 0, \quad (17)$$

$$-3B_1Lr + (-Ap^2 + 2B_1L^2 + C\omega p)\theta = 0. \quad (18)$$

Elimination of  $r$  and  $\theta$  between (17) and (18) gives for the periods

$$\begin{vmatrix} -p^2 + 6B_1/M, & -3B_1L/M, \\ -3B_1L, & -Ap^2 + C\omega p + 2B_1L^2, \end{vmatrix} = 0. \quad (19)$$

It may be remarked that the bi-quadratic equation (19) has four real roots, as is easily seen on noticing the change of sign of the expression on the left-hand side, when for  $p$  is substituted in succession the values  $-\infty$ ,  $-\sqrt{6B_1/M}$ ,  $0$ ,  $\sqrt{6B_1/M}$ ,  $\infty$ . For any one of the four periods, the corresponding ratio of  $r$  to  $\theta$  is given by either (18) or (19).

The most general "free" motion is a linear combination of expressions of the type (16) corresponding to the four periods. To each motion of the type (16), there are two arbitrary constants, viz.,  $r$  and  $\gamma$ . Hence, the general solution involves eight arbitrary constants, which is clearly correct to enable arbitrarily assigned values to be given to  $x$ ,  $\dot{x}$ ,  $y$ ,  $\dot{y}$ ,  $\lambda$ ,  $\dot{\lambda}$ ,  $\mu$ , and  $\dot{\mu}$ . The existence of four real periods  $p$  implies that so far as the "free" or "natural" vibrations are concerned, there is no question of instability.

To complete the solution of equations (8) to (11), we require the "forced" oscillation due to the terms in  $\cos(\omega t + \alpha)$ , *i. e.* due to want of balance in the flywheel. Such an oscillation is obtained by supposing the centre of the flywheel or disk to describe a circle of radius  $R$  with angular velocity  $\omega$  in the same direction as the rotation of the shaft. Such a motion must satisfy

$$x = R \cos(\omega t + \alpha), \quad y = R \sin(\omega t + \alpha), \text{ etc.,}$$

and thus

$$(6B_1/M - \omega^2)R - 3B_1L\Theta/M = \omega^2b,$$

$$-3B_1LR + (2L^2B_1 - A\omega^2 + C\omega^2)\Theta = 0;$$

where  $\Theta$  represents the steady deviation of the free end of the shaft in such a motion. From these equations we find

$$\begin{aligned} \frac{R}{2L^2B_1 - A\omega^2 + C\omega^2} &= \frac{\Theta}{3BL} \\ &= \frac{\omega^2b}{(6B_1/M - \omega^2)(2L^2B_1 - A\omega^2 + C\omega^2) - 9B_1^2L^2/M}. \end{aligned} \quad (20)$$

This agrees with the result (equation 17) given in the author's previous paper (*loc. cit.*) Whirling will take place when

$$(6B_1/M - \omega^2)(2L^2B_1 - A\omega^2 + C\omega^2) - 9B_1^2L^2/M = 0, \quad (21)$$

for in this case the values of  $R$  and  $\Theta$  become infinite. It may be noted that (21) agrees with Chree's result (see *Phil. Mag.*, May 1904, equation (10)). These results accordingly indicate that whirling is the only cause of instability, and provided the internal friction of the shaft prevents the shaft from excessive whirling as the system is speeded up through

the whirling speed, once this speed is left behind, the shaft is as safe as ever. It ought to be mentioned that equation (21), which is a quadratic in  $\omega^2$ , has only one positive root for  $\omega^{2*}$ , so that there is only one whirling speed.

The general solution of equations (8) to (11) is obtained by combining the most general set of free oscillations with the forced circular motion given by (20). In actual practice there will be a certain amount of energy dissipation taking place, and this may be relied on to cause the terms representing the free vibrations ultimately to disappear, leaving the forced motion given by (20) representing the steady state of motion ultimately set up.

It may be noticed that in both the forced motion and in any one of the free motions typified by (16), the path of the centre of gravity  $G$  of the flywheel or disk is a circle concentric with the axis of  $z$ , *i. e.*, the constrained axis. The axis of the shaft is also bent in one plane, which must pass through  $Oz$  and include the centre of gravity  $G$ . These results would not hold, however, for the most general motion involving all the four periods.

It is easy to verify that for speeds below the whirling speed, the centre of gravity  $G$  in the steady motion is always on the opposite side of the bent shaft to the axis  $Oz$ . For speeds above the whirling speed,  $G$  is on the same side of the bent shaft as  $Oz$ , in the steady motion.

#### *Effect of Internal Viscosity Terms on the Motion.*

§ 6. The causes of the various frictional resistances which affect the system will be complex (sound-waves, wind-resistance, viscosity of material of shaft). We shall first assume that at the constrained end of the shaft, arrangements are provided to keep the speed from falling, and for the rest, that friction is effectively represented by linear terms in the rates of variation (relative to the constrained end of the shaft) of the coordinates specifying the configuration of the system. We also assume that the frictional terms will be unaltered if the shaft be considered at rest at the bearing, and the same relative motions take place in the system; this in effect assumes that internal viscosity is the principal factor in the frictional terms.

If we take axes  $O_1\xi$ ,  $O_1\eta$  rotating with the shaft at angular velocity  $\omega$  in a plane parallel to  $xOy$ , the new origin  $O_1$  being on the axis of the shaft at its constrained end (the bearing), we may easily express the above assumptions.

\* Provided, of course, the shaft is long enough, *i. e.*  $L > 3(C - A)/M$ .



Relative to such axes, we have to change the constant  $B_1$  to  $B_1(1 + \alpha D/Dt)$ , where  $D/Dt$  represents differentiation with the moving axes. The constant  $\alpha$  is a constant of viscosity.

The restoring force parallel to  $O_1\xi$  instead of being  $3B_1(2\xi - Ll)$  is now  $3B_1(1 + \alpha D/Dt)(2\xi - Ll)$ . The direction-cosines of the end of the shaft at C relative to the moving axes and the axis of  $z$  are now taken as  $l, m, 1$ . Thus the viscous force  $\Xi$  in the direction  $O_1\xi$  is

$$3B_1\alpha \frac{D}{Dt} (2\xi - Ll).$$

Similarly, parallel to  $O_1\eta$ , we get the viscous force

$$H = 3B_1\alpha \frac{D}{Dt} (2\eta - Lm).$$

Now the viscous force along the *fixed* axis  $Ox$  is

$$X = \Xi \cos \omega t - H \sin \omega t, \quad \dots \quad (22)$$

whilst

$$\xi = x \cos \omega t + y \sin \omega t$$

and

$$\eta = -x \sin \omega t + y \cos \omega t.$$

On evaluating (22), and employing

$$l = \lambda \cos \omega t + \mu \sin \omega t,$$

$$m = -\lambda \sin \omega t + \mu \cos \omega t,$$

we find that

$$X = 6B_1\alpha(\dot{x} + \omega y)/M - 3B_1L\alpha(\dot{\lambda} + \omega\mu)/M. \quad \dots \quad (23)$$

By similar reasoning, the viscous force parallel to the fixed axis  $Oy$  is

$$Y = 6B_1\alpha(\dot{y} - \omega x)/M - 3B_1L\alpha(\dot{\mu} - \omega\lambda)/M. \quad \dots \quad (24)$$

A similar procedure to the above gives the viscous couples, and we are led to the equations of motion

$$\left. \begin{aligned} (D^2 + 6B_1/M)x - 3B_1L\lambda/M + 6B_1\alpha(\dot{x} + \omega y)/M \\ \quad - 3B_1L\alpha(\dot{\lambda} + \omega\mu)/M = b\omega^2 \cos \omega t, \\ (D^2 + 6B_1/M)y - 3B_1L\mu/M + 6B_1\alpha(\dot{y} - \omega x)/M \\ \quad - 3B_1L\alpha(\dot{\mu} - \omega\lambda)/M = b\omega^2 \sin \omega t, \\ -3B_1Lx + (AD^2 + 2B_1L^2)\lambda + C\omega D\mu \\ \quad - 3B_1\alpha L(\dot{x} + \omega y) + 2B_1\alpha L^2(\dot{\lambda} + \omega\mu) = 0, \\ -3B_1Ly + (AD^2 + 2B_1L^2)\mu - C\omega D\lambda \\ \quad - 3B_1\alpha L(\dot{y} - \omega x) + 2B_1\alpha L^2(\dot{\mu} - \omega\lambda) = 0, \end{aligned} \right\} \quad (25)$$

where, as before,  $D$  stands for  $d/dt$ .

These equations can be solved in the usual way, when it will be seen that the effect of internal viscous effects is to modify slightly the periods given by (19) and to introduce damping. It is easy to see, however, that the forced motion is not affected in the slightest, as terms of the type  $(\dot{x} + \omega y)$  then disappear. Hence for whirling, (21) still holds.

For a perfectly balanced shaft we may put  $b=0$ . In such a case, it may appear at first sight that there is a possibility of there being a state of steady motion with  $x$ ,  $y$ ,  $\lambda$ , and  $\mu$  constant. That there is no such possibility may be seen by first wiping out all terms in equations (25) involving  $d/dt$ , and thence showing that the eliminant of the simple linear equations left can never vanish\*.

### *Friction which may prevent Whirling.*

§ 7. Although purely internal elastic (or viscous) friction does not in the slightest degree prevent whirling, yet external friction (*e. g.*, of air) of a suitable type may do so. We shall consider the case of such external frictional effects being dependent linearly on the coordinate velocities, which have no longer to be referred to moving axes. A simple case would be obtained by changing  $B_1$  into  $B_1(1 + \beta d/dt)$ , where  $d/dt$  has now no reference to moving axes. A more general case could be obtained by assuming a dissipation function

$$F = \frac{1}{2}(K_1 \dot{x}^2 + 2K_2 \dot{x}\dot{\lambda} + K_3 \dot{\lambda}^2 + K_4 \dot{y}^2 + 2K_5 \dot{y}\dot{\mu} + K_6 \dot{\mu}^2), \quad (26)$$

such that, *e. g.*, the frictional force parallel to  $Ox$  is  $\partial F/\partial \dot{x}$ , the frictional couple in the plane  $zOx$  is  $\partial F/\partial \dot{\lambda}$ . Without going into the details, it will be noticed that the effect of friction in these cases will be (1) to alter the period equations slightly and thus to give rise to damping of the free vibrations; (2) to prevent the forced oscillation from ever becoming infinite. The general argument for the latter statement will be found in Rayleigh's 'Sound,' vol. i. § 104. Thus friction of the type indicated may enable the system to pass through the critical speed, without disaster occurring.

### *Elastic Yielding at the Bearing.*

§ 8. The effect of elastic yielding at the bearing will now be discussed with the assumption that the forces involved are not sufficient to produce any appreciable movement at the

\* The vanishing of the eliminant would require

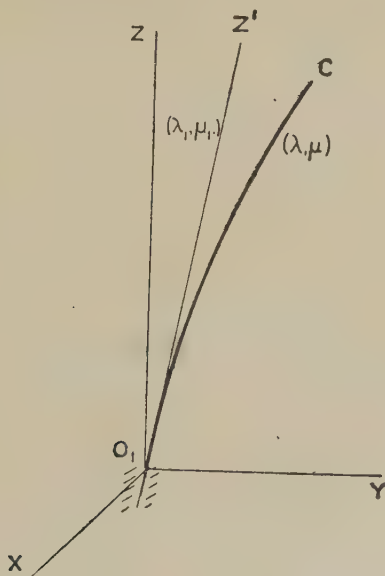
$$17\omega^4\alpha^4 + 2\omega^2\alpha^2 + 1 = 0,$$

which for real values of  $\omega$  can never be satisfied.

centre of the bearing, but only produce small changes in the direction of the *axis* of the bearing. For the sake of generality, this yielding will be taken as different (for the same bending moment) for two directions at right angles to one another.

Referring to fig. 3, let the direction-cosines of the deflected axis of the bearing at  $O_1$  relative to fixed axes be  $\lambda_1, \mu_1, 1$ , and of the axis of the shaft at C be  $\lambda, \mu, 1$ . Relative to  $O_1 z'$ \*,

Fig. 3.



the direction-cosines are  $\lambda_2 (= \lambda - \lambda_1)$ ,  $\mu_2 (= \mu - \mu_1)$ , 1, correct to first order terms. If  $x$  and  $y$  be the coordinates of the centre of the shaft at C relative to the same fixed axes, then relative to  $O_1 z'$ , the coordinates are

$$x_2 = x - L\lambda_1, \quad y_2 = y - L\mu_1.$$

The energy of bending of the shaft is

$$B_1 \{ 3(x_2^2 + y_2^2) - 3L(\lambda_2 x_2 + \mu_2 y_2) + L^2(\lambda_2^2 + \mu_2^2) \}. \quad (27)$$

We shall take the energy of deformation of the bearing as

$$\frac{1}{2}(P\lambda_1^2 + Q\mu_1^2), \quad \dots \dots \dots (28)$$

where the constant  $P$  is not assumed to be equal to  $Q$ , and a term involving  $\lambda_1 \mu_1$  is omitted. The bending moment at  $O_1$  in a plane parallel to  $zO_1x$  is accordingly  $d(\frac{1}{2}P\lambda_1^2)/d\lambda_1 = P\lambda_1$ .

\* The other axes may here be taken as lying in the planes  $zO_1x$ ,  $zO_1y$ .

Similarly, we get  $Q\mu_1$  for the bending moment in a plane parallel to  $zO_1y$ . The restoring force at C parallel to  $O_1x$  is  $3B_1(2x_2 - L\lambda_2)$ , whilst the bending moment at C in the same plane  $zOx$  is  $B_1(-3Lx_2 + 2L^2\lambda_2)$ . Hence the bending moment at  $O_1$  in the same plane ( $zO_1x$ ) is also

$$3B_1(2x_2 - L\lambda_2)L + B_1(-3Lx_2 + 2L^2\lambda_2) = 3B_1Lx_2 - B_1L^2\lambda_2,$$

which must equal  $P\lambda_1$ . Hence

$$P\lambda_1 = 3B_1L(x - L\lambda_1) - B_1L^2(\lambda - \lambda_1),$$

$$\text{or} \quad \lambda_1 = (3B_1Lx - B_1L^2\lambda)/(P + 2B_1L^2), \quad \dots \quad (29)$$

$$\text{Similarly} \quad \mu_1 = (3B_1Ly - B_1L^2\mu)/(Q + 2B_1L^2). \quad \dots \quad (30)$$

Thus the total potential energy is

$$\begin{aligned} V = B_1 \{ & 3[(x - L\lambda_1)^2 + (y - L\mu_1)^2] - 3L[(\lambda - \lambda_1)(x - L\lambda_1) \\ & + (\mu - \mu_1)(y - L\mu_1)] + L^2[(\lambda - \lambda_1)^2 \\ & + (\mu - \mu_1)^2] \} + \frac{1}{2}P\lambda_1^2 + \frac{1}{2}Q\mu_1^2, \quad \dots \quad (31) \end{aligned}$$

where  $\lambda_1$  and  $\mu_1$  are to be replaced by their values (29) and (30) respectively.

$$\text{If we write} \quad B_1L^2/(P + 2B_1L^2) = \alpha,$$

$$\text{and} \quad B_1L^2/(Q + 2B_1L^2) = \beta,$$

where  $\alpha$  and  $\beta$  need not necessarily be small, we find for example :

Restoring force parallel to  $Ox$

$$= 3B_1 \left[ 2x \left( 1 - \frac{3\alpha}{2} \right) - L\lambda(1 - \alpha) \right]. \quad \dots \quad (32)$$

Restoring couple in plane  $zOx$

$$= B_1 \left[ -3Lx(1 - \alpha) + 2L^2\lambda \left( 1 - \frac{\alpha}{2} \right) \right]. \quad \dots \quad (33)$$

The equations of motion of the system now become (neglecting frictional effects)

$$\left. \begin{aligned} \frac{d^2}{dt^2} (x + b \cos \omega t) + 3B_1 \left[ 2x \left( 1 - \frac{3\alpha}{2} \right) - L\lambda(1 - \alpha) \right] / M &= 0, \\ \frac{d^2}{dt^2} (y + b \sin \omega t) + 3B_1 \left[ 2y \left( 1 - \frac{3\beta}{2} \right) - L\mu(1 - \beta) \right] / M &= 0, \\ \frac{d}{dt} (C\lambda\omega - A\dot{\mu}) - B_1 \left[ -3Ly(1 - \beta) + 2L^2\mu \left( 1 - \frac{\beta}{2} \right) \right] &= 0, \\ \frac{d}{dt} (C\mu\omega + A\dot{\lambda}) + B_1 \left[ -3Lx(1 - \alpha) + 2L^2\lambda \left( 1 - \frac{\alpha}{2} \right) \right] &= 0. \end{aligned} \right\} \quad (34)$$



For the natural periods (ignoring the terms in  $b$ ), we find solutions of the type

$$\left. \begin{aligned} x &= r_1 \cos (pt + \gamma), & \lambda &= \theta_1 \cos (pt + \gamma), \\ y &= r_2 \sin (pt + \gamma), & \mu &= \theta_2 \sin (pt + \gamma). \end{aligned} \right\} \quad (35)$$

Of the four constants ( $r_1, r_2, \theta_1, \theta_2$ ) only one is arbitrary, the others being given by (34). For the values of  $p$ , we have the equation

$$\Delta(p) =$$

$$\begin{vmatrix} -p^2 + \frac{6B_1}{M} \left(1 - \frac{3\alpha}{2}\right) & 0 & -\frac{3B_1L}{M} (1 - \alpha) & 0 \\ 0 & -p^2 + \frac{6B_1}{M} \left(1 - \frac{3\beta}{2}\right) & 0 & -\frac{3B_1L}{M} (1 - \beta) \\ 0 & 3B_1L(1 - \beta) & -C\omega p & Ap^2 - 2B_1L^2 \left(1 - \frac{\beta}{2}\right) \\ 3B_1L(1 - \alpha) & 0 & Ap^2 - 2B_1L^2 \left(1 - \frac{\alpha}{2}\right) & -C\omega p \end{vmatrix} = 0. \quad (36)$$

This is a biquadratic in  $p^2$ , and for small values of  $\alpha$  and  $\beta$ , the values of  $p$  will differ little from those given by (19). We may take it that the natural oscillations will ultimately be damped out. The steady forced oscillation left can be obtained by assuming

$$\left. \begin{aligned} x &= R_1 \cos \omega t, & \lambda &= \Theta_1 \cos \omega t, \\ y &= R_2 \sin \omega t, & \mu &= \Theta_2 \sin \omega t. \end{aligned} \right\} \quad (37)$$

Substituting these values in equations (34), we find

$$\left. \begin{aligned} R_1 \left\{ 6B_1 \left(1 - \frac{3\alpha}{2}\right) / M - \omega^2 \right\} - 3B_1L\Theta_1(1 - \alpha)/M &= \omega^2 b, \\ R_2 \left\{ 6B_1 \left(1 - \frac{3\beta}{2}\right) / M - \omega^2 \right\} - 3B_1L\Theta_2(1 - \beta)/M &= \omega^2 b, \\ 3B_1L(1 - \beta)R_2 - C\omega^2\Theta_1 + \Theta_2 \left\{ A\omega^2 - 2B_1L^2 \left(1 - \frac{\beta}{2}\right) \right\} &= 0, \\ 3B_1L(1 - \alpha)R_1 - C\omega^2\Theta_2 + \Theta_1 \left\{ A\omega^2 - 2B_1L^2 \left(1 - \frac{\alpha}{2}\right) \right\} &= 0. \end{aligned} \right\} \quad (38)$$

The four equations (38) determine  $R_1, R_2, \Theta_1$ , and  $\Theta_2$ . If the determinant arising out of the left-hand sides of these equations (on elimination of  $R_1, R_2, \Theta_1$ , and  $\Theta_2$ ) vanishes, the amplitudes  $R_1$ , etc. become infinite, and we get whirling. The whirling equation is easily seen to be the result of substituting  $\omega$  for  $p$  in the equation (36).

The path of the free extremity C of the axis of the shaft is the ellipse

$$x^2/R_1^2 + y^2/R_2^2 = 1. \quad \dots \quad (39)$$

This elliptical path is to be expected when the constrained end of the axis is not very rigidly held.

§ 9. We shall now discuss at greater length some of the points involved in § 8.

With regard to the period equation (36), it is possible to show that this equation always has real positive roots for  $p^2$ , and hence (except at a whirling speed) there is no question of instability. To prove this statement, we follow the method outlined by Routh ('Rigid Dynamics, vol. ii. p. 50 (1905)).

We write (36) in the form

$$\Delta = \begin{vmatrix} -p^2 + \frac{6B_1}{M} \left(1 - \frac{3\alpha}{2}\right), & 0, & -\frac{3B_1L}{M}(1-\alpha), & 0 \\ 0, & -p^2 + \frac{6B_1}{M} \left(1 - \frac{3\beta}{2}\right), & 0, & -\frac{3B_1L}{M}(1-\beta) \\ -3B_1L(1-\alpha), & 0, & -Ap^2 + B_1L^2 \left(1 - \frac{\alpha}{2}\right), & C\omega p \\ 0, & -3B_1L(1-\beta), & C\omega p, & -Ap^2 + 2B_1L^2 \left(1 - \frac{\beta}{2}\right) \end{vmatrix} \\ = 0. \quad \dots \quad (36 a)$$

and denote by  $\Delta_1$  the first minor obtained by deleting the first row and the first column. The second minor obtained by deleting the first row and first column of  $\Delta_1$  may be denoted by  $\Delta_2$ . We proceed similarly for  $\Delta_3$ , which will be  $-Ap^2 + 2B_1L^2 \left(1 - \frac{\beta}{2}\right)$ . We also let  $\Delta_4 = 1$ . Each of the determinants referred to will be a function of  $p^2$ , and clearly when  $p^2 = \infty$ , we shall have

$$\Delta = \infty, \Delta_1 = -\infty, \Delta_2 = \infty, \Delta_3 = -\infty, \Delta_4 = 1.$$

In this series, we have four changes of sign.

We now consider the values of the determinants when we take  $p^2 = 0$ . In this case, we find

$$\Delta = \begin{vmatrix} \frac{6B_1}{M} \left(1 - \frac{3\alpha}{2}\right), & -\frac{3B_1L}{M}(1-\alpha) \\ -3B_1L(1-\alpha), & 2B_1L^2 \left(1 - \frac{\alpha}{2}\right) \end{vmatrix} \begin{vmatrix} \frac{6B_1}{M} \left(1 - \frac{3\beta}{2}\right), & -\frac{3B_1L}{M}(1-\beta) \\ -3B_1L(1-\beta), & 2B_1L^2 \left(1 - \frac{\beta}{2}\right) \end{vmatrix} \dots \quad (40)$$

Hence (assuming that  $\alpha$  and  $\beta$  are both positive and less \* than  $\frac{2}{3}$ ) we get  $\Delta$  positive. We also find

$$\Delta_1 = \frac{3B_1^3 L^4}{M} \left(1 - \frac{\alpha}{2}\right) (1 - 2\beta), \quad . \quad . \quad (41)$$

which is also positive, provided  $\beta < \frac{1}{2}$ . Similarly,  $\Delta_2$ ,  $\Delta_3$ , and  $\Delta_4$  are positive, and there are no changes in sign in the series  $\Delta$ ,  $\Delta_1$ ,  $\Delta_2$ ,  $\Delta_3$ , and  $\Delta_4$ . In consequence, in going from  $p^2=0$  to  $p^2=\infty$ , four changes of sign are gained in the series  $\Delta$ ,  $\Delta_1$ , etc., and the roots (in  $p^2$ ) of the bi-quadratic (36a) all lie between 0 and  $\infty$ . Hence  $p$  is always real.

We proceed to discuss the question of whirling. Solving equations (38) in the usual way for  $R_1$  (say), we find

$$\begin{aligned} -\frac{R_1 \bar{\Delta}}{\omega^2 b} = & \left\{ \frac{6B_1}{M} \left(1 - \frac{3\beta}{2}\right) - \omega^2 \right\} \\ & \left\{ C^2 \omega^4 - \left[ A\omega^2 - 2B_1 L^2 \left(1 - \frac{\alpha}{2}\right) \right] \left[ A\omega^2 - 2B_1 L^2 \left(1 - \frac{\beta}{2}\right) \right] \right\} \\ & + \frac{9B_1^2 L^2}{M} (1-\alpha)(1-\beta) C\omega^2 - \frac{9B_1^2 L^2}{M} (1-\beta)^2 \\ & \left[ A\omega^2 - 2B_1 L^2 \left(1 - \frac{\alpha}{2}\right) \right], \quad . \quad . \quad (42) \end{aligned}$$

where  $\bar{\Delta}$  is the determinant obtained by putting  $p^2=\omega^2$  in (36a). It may be remarked that if  $\alpha$  and  $\beta$  are not equal, then  $\bar{\Delta}$  is not capable of being split into factors, and (42) cannot be made simpler in form. If, however, we take  $\alpha=\beta$ , we find

$$\begin{aligned} -\frac{R_1 \bar{\Delta}}{\omega^2 b} = & \left\{ C\omega^2 - \left[ A\omega^2 - 2B_1 L^2 \left(1 - \frac{\alpha}{2}\right) \right] \right\} \cdot \\ & \cdot \left\{ \left[ \frac{6B_1}{M} \left(1 - \frac{3\alpha}{2}\right) - \omega^2 \right] \left[ C\omega^2 + A\omega^2 - 2B_1 L^2 \left(1 - \frac{\alpha}{2}\right) \right] \right. \\ & \left. + \frac{9B_1^2 L^2}{M} (1-\alpha)^2 \right\}, \quad . \quad . \quad (43) \end{aligned}$$

\* It is actually necessary for the complete argument to restrict both  $\alpha$  and  $\beta$  to values less than  $\frac{1}{2}$ . The definitions of  $\alpha$  and  $\beta$  in § 8 show that these restrictions are satisfied.

whilst

$$\Delta = - \left\{ \left[ \frac{6B_1}{M} \left( 1 - \frac{3\alpha}{2} \right) - \omega^2 \right] \left[ C\omega^2 - A\omega^2 + 2B_1L^2 \left( 1 - \frac{\alpha}{2} \right) \right] - \frac{9B_1^2L^2}{M} (1-\alpha)^2 \right\} \\ \cdot \left\{ \left[ \frac{6B_1}{M} \left( 1 - \frac{3\alpha}{2} \right) - \omega^2 \right] \left[ C\omega^2 + A\omega^2 - 2B_1L^2 \left( 1 - \frac{\alpha}{2} \right) \right] + \frac{9B_1^2L^2}{M} (1-\alpha)^2 \right\} \dots \dots (44)$$

Thus the second line factor of (44) disappears in the expression for  $R_1$ , and we get a result agreeing with (20) when  $\alpha=0$ .

If  $\alpha$  and  $\beta$  are not equal, we cannot get rid of a factor in this way, and whirling may occur whenever  $\Delta=0$ . If we denote the minors obtained by deleting from  $\bar{\Delta}$  the first row and column, first two rows and columns, etc. by  $\bar{\Delta}_1, \bar{\Delta}_2$ , etc., exactly as above, we find when  $\omega^2=\infty$ , that (since  $C>A$ )

$$\bar{\Delta} = -\infty, \bar{\Delta}_1 = +\infty, \bar{\Delta}_2 = -\infty, \bar{\Delta}_3 = -\infty, \bar{\Delta}_4 = 1.$$

When we put  $\omega^2=0$ , we find

$$\bar{\Delta} = \begin{vmatrix} \frac{6B_1}{M} \left( 1 - \frac{3\alpha}{2} \right), & \frac{3B_1L}{M} (1-\alpha) \\ 3B_1L(1-\alpha), & 2B_1L^2 \left( 1 - \frac{\alpha}{2} \right) \end{vmatrix} \begin{vmatrix} \frac{6B_1}{M} \left( 1 - \frac{3\beta}{2} \right), & \frac{3B_1L}{M} (1-\beta) \\ 3B_1L(1-\beta), & 2B_1L^2 \left( 1 - \frac{\beta}{2} \right) \end{vmatrix}, \dots \dots (45)$$

and is accordingly positive. We also find

$$\bar{\Delta}_1 = -\frac{3B_1^3L^4}{M} \left( 1 - \frac{\alpha}{2} \right) (1-2\beta), \dots \dots (46)$$

and is thus positive. Clearly also  $\bar{\Delta}_2$  is positive, whilst  $\bar{\Delta}_3$  is positive and  $\bar{\Delta}_4$  is unity. Thus in going from  $\omega^2=0$  to  $\omega^2=\infty$ , we gain three\* changes of sign, and there are

\* There are at least three whirling speeds, but the argument really shows that there may be three, five, or any odd number ( $>3$ ) whirling speeds. The biquadratic in  $\omega^2$  (viz.,  $\bar{\Delta}=0$ ) can, however, only have four roots; hence we settle once and for all that there are three positive roots only for  $\omega^2$ , and hence three real whirling speeds.

For this important point I am indebted to Prof. Orr, who observes that Routh's argument ('Rigid Dynamics,' *loc. cit.*) depends upon  $n$  variations of sign being lost or gained between  $p^2=-\infty$  and  $p^2=+\infty$ , for a determinant of the  $n$ th order which equated to zero gives an equation of the  $n$ th degree in  $p^2$ . If (as in our  $\bar{\Delta}=0$ ),  $n$  variations of sign are not lost or gained (*i. e.* in our case  $n=4$ ) between  $p^2=-\infty$  and  $p^2=+\infty$  (in our whirling equation  $\bar{\Delta}=0$ ,  $p^2$  is  $\omega^2$ ), Routh's argument must be modified in some such way as is indicated at the beginning of this footnote.



accordingly three real positive roots for  $\omega^2$ , and hence *three* whirling speeds. With a *circular* motion of the point C, however, there is only one whirling speed, because (as has already been pointed out) for  $\alpha=\beta$ , the numerator in the expression for  $R_1$  vanishes for two of the speeds just given by the general theory. The two speeds referred to will be given by

$$\left[ \frac{6B_1}{M} \left( 1 - \frac{3\alpha}{2} \right) - \omega^2 \right] \left[ C\omega^2 + A\omega^2 - 2B_1L^2 \left( 1 - \frac{\alpha}{2} \right) \right] + \frac{9B_1^2L^2}{M} (1-\alpha)^2 = 0 \quad (47)$$

If we take  $\alpha=0$ , we get

$$\omega^2 = B_1 \left( \frac{L^2}{C+A} + \frac{3}{M} \right) \pm B_1 \sqrt{\left( \frac{L^2}{C+A} + \frac{3}{M} \right)^2 - \frac{3L^2}{M(C+A)}}, \quad (48)$$

and these will give close approximations to the new critical speeds involved when the shaft is fairly rigidly held.

As a numerical example, we may take  $C=2A$ ,  $L^2=15A/M$ . We then find the whirling speed given by (21) is obtained from  $\omega^2 = (\sqrt{189}-12)B_1/M$ . The two whirling speeds introduced by the imperfect rigidity of the bearing would approximately be given by  $\omega^2 = B_1/M$  and  $15B_1/M$ . The tendency to whirl at these latter speeds might easily be masked by the effects of friction, however.

### *The Effect of Slight Initial Distortion of Shaft.*

§ 10. If the axis of the shaft be not originally a perfect straight line, we may proceed as follows (the bearing is supposed to be rigid).

We still suppose that the coordinates of the point C for the free end of the shaft are  $(x, y)$  relative to fixed axes. The potential energy is, however, now to be taken as expression (5) with  $x-\bar{x}$ ,  $y-\bar{y}$ ,  $\lambda-\bar{\lambda}$ ,  $\mu-\bar{\mu}$ , substituted respectively for  $x, y, \lambda, \mu$ . Here  $\bar{x}$ ,  $\bar{y}$ ,  $\bar{\lambda}$ ,  $\bar{\mu}$ , represent respectively the initial small values of  $x, y, \lambda, \mu$ , due to the slight distortion. The initial bending need not be confined to one plane, and we may suppose

$$\left. \begin{aligned} \bar{x} &= b_1 \cos(\omega t + \alpha_1), & \bar{y} &= b_1 \sin(\omega t + \alpha_1), \\ \bar{\lambda} &= \theta_1 \cos(\omega t + \beta_1), & \bar{\mu} &= \theta_1 \sin(\omega t + \beta_1), \end{aligned} \right\}, \quad (49)$$

where  $b, \theta_1, \alpha_1$ , and  $\beta_1$  are constants.

Instead of equations (8) to (11), we get

$$\left. \begin{aligned} \ddot{x} + 6B_1x/M - 3B_1L\lambda/M &= \omega^2b \cos \omega t \\ &+ 3B_1[2b_1 \cos(\omega t + \alpha_1) - L\theta_1 \cos(\omega t + \beta_1)]/M, \\ \ddot{y} + 6B_1y/M - 3B_1L\mu/M &= \omega^2b \sin \omega t \\ &+ 3B_1[2b_1 \sin(\omega t + \alpha_1) - L\theta_1 \sin(\omega t + \beta_1)]/M, \\ C\omega\dot{\lambda} - A\ddot{\mu} - B_1(-3Ly + 2L^2\mu) \\ &= B_1[-2L^2\theta_1 \sin(\omega t + \beta_1) + 3Lb_1 \sin(\omega t + \alpha_1)], \\ C\omega\dot{\mu} + A\ddot{\lambda} + B_1(-3Lx + 2L^2\lambda) \\ &= B_1[2L^2\theta_1 \cos(\omega t + \beta_1) - 3Lb_1 \cos(\omega t + \alpha_1)]. \end{aligned} \right\} \quad (50)$$

The free periods will be exactly the same as those given in § 5. The forced motion will involve, for each of the four coordinates, simple harmonic terms of the type  $\cos \omega t$  and  $\sin \omega t$ . We may, in fact, assume

$$\left. \begin{aligned} x &= A_1 \cos \omega t + A_2 \sin \omega t, \\ y &= A_1 \sin \omega t - A_2 \cos \omega t, \\ \lambda &= A_3 \cos \omega t + A_4 \sin \omega t, \\ \mu &= A_3 \sin \omega t - A_4 \cos \omega t. \end{aligned} \right\} \quad \dots \quad (51)$$

If we substitute these values in either the first and third, or the second and fourth of equations (50), and equate the terms in  $\cos \omega t$  and  $\sin \omega t$  together for both sides of each of these equations, we get four equations for determining  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_4$ , viz.,

$$\left. \begin{aligned} A_1 \left( -\omega^2 + \frac{6B_1}{M} \right) - \frac{3B_1L}{M} A_3 \\ &= \omega^2b + \frac{3B_1}{M} (2b_1 \cos \alpha_1 - L\theta_1 \cos \beta_1), \\ A_2 \left( -\omega^2 + \frac{6B_1}{M} \right) - \frac{3B_1L}{M} A_4 \\ &= -\frac{3B_1}{M} (2b_1 \sin \alpha_1 - L\theta_1 \sin \beta_1), \\ -3B_1LA_1 + A_3(C\omega^2 - A\omega^2 + 2B_1L^2) \\ &= 2B_1L^2\theta_1 \cos \beta_1 - 3B_1Lb_1 \cos \alpha_1, \\ -3B_1LA_2 + A_4(C\omega^2 - A\omega^2 + 2B_1L^2) \\ &= -2B_1L^2\theta_1 \sin \beta_1 + 3B_1Lb_1 \sin \alpha_1. \end{aligned} \right\} \quad \dots \quad (52)$$

In the denominators of the expressions for each of  $A_1$ ,  $A_2$ ,

$A_3$ , and  $A_4$ , will appear (since  $A_1$  and  $A_3$  are independent of  $A_2$  and  $A_4$ )

$$\Delta_1 = \begin{vmatrix} -\omega^2 + \frac{6B_1}{M}, & -\frac{3B_1L}{M} \\ -3B_1L, & C\omega^2 - A\omega^2 + 2B_1L^2 \end{vmatrix} \quad (53)$$

Thus the condition for whirling is

$$\Delta_1 = 0, \quad (54)$$

which is precisely the same as equation (21). Thus whether we consider whirling as arising from the forced motion due to an eccentric load, or to the forced motion due to a slightly deformed shaft initially, the same result for the whirling speed is obtained.

It is quite clear from (51) that in the forced motion the path of C is a circle concentric with Oz, for

$$x^2 + y^2 = A_1^2 + A_2^2.$$

The points C and G, and the axis Oz need not now necessarily lie in the same plane.

### Conclusion.

§ 11. In the present paper, the author has endeavoured to indicate the kinematics of the motion of the free end of the shaft assuming negligible torsional effects. The general equations of motion have then been deduced for an eccentrically loaded disk or flywheel, leading to the conclusion that whirling may be regarded as due to the amplitude of the steady forced oscillation of C round Oz becoming theoretically infinite at a certain speed. Alternatively, whirling (even for a symmetrically mounted flywheel) may be regarded as having a similar origin, on account of slight deviations of the centre line of the shaft from the straight line initially.

The author has endeavoured to show that elastic viscosity (in the shaft itself) cannot prevent whirling in the slightest degree, but that external friction may enable the system to pass safely through what would be a whirling speed without such friction.

The effect of elastic yielding at the bearing has been investigated both for symmetrical and unsymmetrical elasticity. The curious result is obtained that for unsymmetrical elasticity or elliptic motion there are three whirling speeds (neglecting friction). Apart from whirling, there is no question of instability, however. The two extra whirling

speeds are traced down to an effect which vanishes when the motion becomes circular. It is possible that external friction may mask these additional speeds of whirling.

Lastly, the author takes this opportunity of acknowledging his deep indebtedness to Prof. W. McF. Orr, for his many valuable suggestions and criticisms whilst the material of this paper has been thrashed out. But for him this paper could not have been written in anything like its present form.

LXVII *Note on Mr. Lees' Paper on the Whirling of an Overhung Shaft.* By Prof. W. McF. ORR, D.Sc., F.R.S.\*

THIS note originated from a correspondence with Mr. Lees. My objection to his use of Lagrange's equation was by no means the only point about which he showed me that I was in error.

§1. *If the three moments of inertia are unequal the motion is unstable at high speeds.*

In connexion with the problem discussed by Mr. Lees it occurred to me to test for stability in case the three moments of inertia, A, B, C, of the pulley relative to its centre of mass are all unequal. There are then two "whirling" speeds, each being that for the same shaft loaded with a symmetrical pulley, having moments A, A, C in one case and B, B, C in the other. And, to my surprise, I found that the "free" motion for a perfectly centred pulley is exponentially unstable if either of these speeds is exceeded. This may explain the fact, of which I have been informed by my former colleague, Mr. H. H. Jeffcott †, that propellers (of aeroplanes) having two blades show evidence of instability.

Although I solved the problem first in Lees' coordinates, apparently the shortest way of doing so is to follow Routh's treatment of the unsymmetrical top (Rigid Dynamics, 4th edn., ii. Arts. 211, 212), thus taking, as two of the variables, not Lees'  $\lambda, \mu$  (§4), but the cosines of the angles which the fixed axis of the bearing makes with the principal axes GA, GB, of the pulley. This is virtually the same as

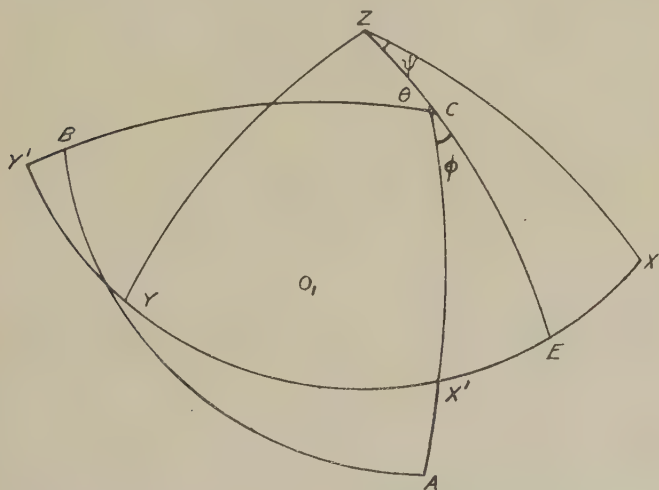
\* Communicated by the Author.

† I may refer any one who desires a satisfactory explanation of the dynamical cause of "whirling," as free as seems possible from mathematical difficulties, to a paper by Jeffcott:—"The Lateral Vibration of Loaded Shafts," Phil. Mag. xxxvii. March 1919.



referring to axes rotating with angular velocity which is approximately constant and equal to that of the shaft.

In the diagram, which is Routh's (*l. c. i. p.* 217) with some additional letters, the origin is Lees'  $O_1$ , the fixed point on axis of the shaft; the axes  $O_1X$ ,  $O_1Y$ ,  $O_1Z$  are fixed, the last being the axis of the bearing;  $O_1A$ ,  $O_1B$ ,  $O_1C$  are parallel to the principal axes of the pulley at its C.G., the last being perpendicular to its plane. As  $\angle X'E$  is approximately  $\phi$ , the axes  $O_1X'$ ,  $O_1Y'$  rotate with approximately constant velocity  $\omega$ , that of the shaft. Relative to the fixed axes let the direction-cosines of  $O_1A$  be  $\lambda_1, \mu_1, \nu_1$ ; of  $O_1B$ ,  $\lambda_2, \mu_2, \nu_2$ ; of  $O_1C$ ,  $\lambda_3, \mu_3, \nu_3$ . Relative to  $O_1X'$ ,  $O_1Y'$ , let  $x', y',$  be the



coordinates of the foot of the perpendicular from the C.G. of the pulley on the bent axis of the shaft, and  $\lambda', \mu'$ , the direction-cosines of  $O_1C$ .

In the case of the pulley excentrically mounted, in estimating the restoring forces and couples, (and these are known correctly to only the first order of small quantities), it is immaterial, to the first order, whether we suppose the forces applied at the C.G. of the pulley, (Lees'  $G$ ), or at the corresponding point on the axis of the shaft, (Lees'  $C$ ). We have restoring forces  $3 B_1(2x' - L\lambda')$  parallel to  $X'O_1$ ,  $3 B_1(2y' - L\mu')$  parallel to  $Y'O_1$ , and restoring couples  $B_1(2L^2\mu' - 3Ly')$  about  $O_1X'$ , tending to diminish  $\mu'$ ,  $B_1(2L^2\lambda' - 3Lx')$  about  $O_1Y'$ , tending to diminish  $\lambda'$ .

But  $\lambda'$ , or  $\cos CX'$ , is  $\sin AX'$ , and this is approximately  $-\cos AZ$  or  $-\nu_1$ , since, in triangle  $AX'Z$ ,  $X'Z$  is  $\pi/2$  and  $ZX'A$  is nearly  $\pi$ . Similarly,  $\mu' \doteq -\nu_2$ .

Again,  $O_1X'$ ,  $O_1A$  nearly coincide, and so do  $O_1Y'$ ,  $O_1B$ ; therefore, to the first order, the moment of restoring forces round  $GA$  equals that round  $GX'$ , and similarly for  $GB$ ,  $GY'$ .

Thus, if the pulley is "out of balance," we have, by resolving parallel to  $GX'$ ,  $GY'$ , and taking moments round  $GA$ ,  $GB$ , the equations of motion, correct to the first order,

$$M(\ddot{x}' - 2\omega\dot{y}' - \omega^2x') + 3B_1(2x' + L\nu_1) = Mb\omega^2 \cos(\omega t + \alpha), \quad (1)$$

$$M(\ddot{y}' + 2\omega\dot{x}' - \omega^2y') + 3B_1(2y' + L\nu_2) = Mb\omega^2 \sin(\omega t + \alpha). \quad (2)$$

$$A\dot{\omega}_1 - (B - C)\omega\omega_2 + B_1(3Ly' + 2L^2\nu_2) = 0, \quad \dots \quad (3)$$

$$B\dot{\omega}_2 - (C - A)\omega\omega_1 - B_1(3Lx' + 2L^2\nu_1) = 0, \quad \dots \quad (4)$$

$\omega_1$ ,  $\omega_2$ , being velocities about  $GA$ ,  $GB$ .

Also, as  $\nu_1$ ,  $\nu_2$ ,  $\nu_3$  are direction-cosines with respect to the pulley of a line fixed in space, we have the kinematic equations, correct to the first order,

$$\dot{\nu}_1 - \nu_2\omega + \omega_2 = 0, \quad \dots \quad (5)$$

$$\dot{\nu}_2 - \omega_1 + \nu_1\omega = 0. \quad \dots \quad (6)$$

In these, of course,  $\omega_3$ , the velocity of the pulley round  $GC$ , has been replaced by  $\omega$ , the velocity of rotation of the shaft.

Eliminating  $\omega_1$ ,  $\omega_2$ , from (3)–(6), we obtain

$$A\ddot{\nu}_2 + (A + B - C)\omega\dot{\nu}_1 + (C - B)\omega^2\nu_2 + B_1(3Ly' + 2L^2\nu_2) = 0, \quad \dots \quad (7)$$

$$B\ddot{\nu}_1 - (A + B - C)\omega\dot{\nu}_2 + (C - A)\omega^2\nu_1 + B_1(3Lx' + 2L^2\nu_1) = 0, \quad \dots \quad (8)$$

For the Complementary Function, in other words for the "free" vibrations of a perfectly centred, but unsymmetrical, pulley, there are evidently solutions of the form

$$x' = X \cos(pt + \gamma), \quad y' = Y \sin(pt + \gamma), \quad \nu_1 = N_1 \cos(pt + \gamma), \\ \nu_2 = N_2 \sin(pt + \gamma), \quad \dots \quad (9)$$

where  $p$ ,  $\gamma$ ,  $X$ ,  $Y$ ,  $N_1$ ,  $N_2$ , are constants.

With these substitutions the equations for the C.F. become

$$\{2B_1L^2 + (C - B)\omega^2 - Ap^2\}N_2 - (A + B - C)\omega pN_1 \\ + 3B_1LY = 0, \quad \dots \quad (10)$$

$$\{2B_1L^2 + (C - A)\omega^2 - Bp^2\}N_1 - (A + B - C)\omega pN_2 \\ + 3B_1LX = 0, \quad \dots \quad (11)$$

$$3B_1LN_1 + \{6B_1 - M(\omega^2 + p^2)\}X - 2M\omega pY = 0, \quad \dots \quad (12)$$

$$3B_1LN_2 + \{6B_1 - M(\omega^2 + p^2)\}Y - 2M\omega pX = 0, \quad \dots \quad (13)$$

The equation giving the "speed,"  $p$ , may, therefore, be written

$$\begin{vmatrix} 2B_1L^2 + (C-B)\omega^2 - Ap^2, & 3B_1L, & -(A+B-C)\omega p, & 0 \\ 3B_1L, & 6B_1 - M(\omega^2 + p^2), & 0, & -2M\omega p \\ -(A+B-C)\omega p, & 0, & 2B_1L^2 + (C-A)\omega^2 - Bp^2, & 3B_1L \\ 0, & -2M\omega p, & 3B_1L, & 6B_1 - M(\omega^2 + p^2) \end{vmatrix} = 0. \quad (14)$$

In the treatment followed here a "whirling" speed of rotation presents itself as a value of  $\omega$  for which (14) is satisfied by  $p=0$ . It is seen, therefore, that there are two real "whirling" speeds, each being that for the same shaft loaded with a pulley of mass  $M$ , but with moments  $A, A, C$ , in one case and  $B, B, C$  in the other. It is easily verified that, if  $A < B$ , the latter speed is the lower.

Proceeding to examine the nature of the "free" oscillations for other speeds of rotation, I follow Routh's argument and notation (*l. c.*, ii. Arts. 57 *et seq.*). When  $p^2 = -\infty$  the signs of  $\Delta, \Delta_1, \Delta_2, \Delta_3, 1$ , are all positive; when  $p^2 = +\infty$  they are alternately positive and negative; so that, while  $p^2$  passes from  $-\infty$  to  $+\infty$ , four, (the greatest number possible), changes of sign are gained. Therefore the values of  $p^2$  given by (14) are all real, and, also, the number between  $p^2 = \alpha$  and  $p^2 = \beta$ , where  $\alpha < \beta$ , is the same as the number of changes of sign in the above series which is gained as  $p^2$  increases from  $\alpha$  to  $\beta$ .

Firstly, suppose  $\omega$  below both "whirling" speeds. Then, when  $p^2 = 0$ ,  $\Delta_3$  is positive, since each "whirling" speed is less than the value of  $\omega$  given by  $\Delta_3 = 0$ ;  $\Delta_2$  also is positive, since the "whirling" speed for the shaft carrying a pulley of moments  $A, A, C$  is given by  $\Delta_2 = 0$ , and  $\Delta_2$  is positive for lower values of  $\omega$ ;  $\Delta_1$  also is positive, being equal to  $(6B_1 - M\omega^2)\Delta_2$ ; and  $\Delta$  is positive, being the product of  $\Delta_2$  and a determinant which differs from  $\Delta_2$  in having  $A$  replaced by  $B$  and which, therefore, is positive. Thus there is no alternation of sign in the series  $\Delta, \Delta_1, \Delta_2, \Delta_3, 1$ . Consequently, all four values of  $p^2$  given by (14) are positive.

Secondly, suppose that,  $A$  being less than  $B$ ,  $\omega$  exceeds the "whirling" speed for  $B, B, C$ , but is less than that for  $A, A, C$ . When  $p^2$  is zero  $\Delta_3, \Delta_2, \Delta_1$  are again positive; but  $\Delta$  is negative, since it is the product of  $\Delta_2$  and a similar determinant which is now negative, however. Thus there is one alternation of sign in the series of determinants.

Consequently (14) gives one negative value of  $p^2$  and three positive.

Thirdly, suppose that  $\omega$  exceeds both "whirling" speeds but is less than  $(6B_1/M)^{\frac{1}{2}}$ . When  $p^2$  is zero,  $\Delta_3$  is again positive, but  $\Delta_2$  is negative,  $\Delta_1$  is negative, and  $\Delta$  is positive, being the product of  $\Delta_2$  and a similar determinant, now also negative. Thus there are two alternations of sign in the series. Consequently (14) now gives two negative values of  $p^2$  and two positive.

Finally, suppose  $\omega^2 > 6B_1/M$ . When  $p^2$  is zero,  $\Delta_3$  is now negative,  $\Delta_2$  is negative, while  $\Delta_1$ ,  $\Delta$  are positive. Thus two alternations of sign again occur. And, therefore, again two values of  $p^2$  are negative and two positive.

The motion is, therefore, exponentially unstable if the lower "whirling" speed is exceeded.

### § 2. *An unexplained phenomenon in a certain whirling shaft.*

I would draw attention to a phenomenon, which I cannot explain, observed in a certain whirling shaft. A model of an actual shaft was mounted by Mr. (then Prof.) Jeffcott in the Royal College of Science for Ireland and driven electrically. It had three bearings, one at each end, the other placed unsymmetrically; it carried several pulleys. Its "whirling" speed was of the order of 25 per sec.; but the amplitude of its vibrations varied regularly with a period having order of magnitude one sec.

I did not regard the vibrations of the shaft as "small"; the greatest displacement of its axis from the axis of the bearings was of the order of magnitude of the radius.

The vibrations were elliptical, not circular; Lees explains this by unsymmetrical elasticity in the bearings (see his paper, § 8, p. 698); I believe this explanation to be correct.

### § 3. *A flaw in the usual proof of stability of a system.*

In connexion with discussions of the stability of equilibrium or motion of systems in general, I have elsewhere (Proc. Royal Irish Acad. xxvii. A, 1909, pp. 12, 13, 32-36, 75) objected to the orthodox argument, that, when there is more than one coordinate, if the fundamental free disturbances are truly harmonic, or exponentially damped, then the state is certainly stable. This inference cannot be justified mathematically, even should it agree with fact in the vast majority of cases. A good deal, of course, turns on the precise meaning given to the word "stability." And the



mathematician is inclined to give it a definition which facilitates him in giving an absolute criterion; the engineer is more concerned with "safety" than with "stability." At any rate, a disturbance in which the displacement of each particle is the sum of two or more terms, each of type harmonic or damped harmonic, may be such that at some instant both the displacements and the velocities are simultaneously very great compared with their respective initial values.

I have, indeed, (*l. c.*), put this forward as an explanation of the observed instability of the motion of liquids\* in certain cases. My view, however, did not commend itself to the late Lord Rayleigh.

#### § 4. Absolute proofs of stability in Lees' problem.

In the problems under discussion here perfectly satisfactory proofs of stability † at all speeds with the symmetrical pulley, and under the lower "whirling" speed with the unsymmetrical, may be obtained as follows.

For the symmetrical case on multiplying Lees' equations (12)-(15) respectively by  $\dot{x}$ ,  $\dot{y}$ ,  $\dot{\lambda}$ ,  $\dot{\mu}$ , adding, and integrating, there results

$$\frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}A(\dot{\lambda}^2 + \dot{\mu}^2) + B_1\{3(x^2 + y^2) - 3L(\lambda x + \mu y) + L^2(\lambda^2 + \mu^2)\} = \text{constant}. \quad (15)$$

The terms in  $B_1$  are essentially positive, being, in fact, the elastic potential energy. The equation is not, however, the energy equation.

Consequently, if the steady motion receives a disturbance in which the constant has a small value  $C$ , the shaft can never bend so much that the terms in  $B_1$  alone exceed  $C$ , nor can the value of  $\dot{x}$ ,  $\dot{y}$ ,  $\dot{\lambda}$ , or  $\dot{\mu}$ , exceed a definite small limit.

In the unsymmetrical case the equations of motion in Lees' coordinates give no such simple equation as (15). But, if my equations of motion, (1), (2), (7), (8), with the first two modified by replacing the right-hand member by zero, are multiplied respectively by  $\dot{x}'$ ,  $\dot{y}'$ ,  $\dot{\nu}_2$ ,  $\dot{\nu}_1$ , and added,

\* An important paper, just published, by G. I. Taylor, "Stability of a viscous fluid contained between two rotating cylinders," *Phil. Trans. A.* 612, appears to prove that, under certain conditions, the fundamental free disturbances are exponentially unstable, and thus to give a simpler explanation.

† That is, taking, as usual, equations of motion which are correct only to the first order of small quantities.

the resulting equation gives, on integration,

$$\begin{aligned} \frac{1}{2}M(\dot{x}'^2 + \dot{y}'^2) + \frac{1}{2}(A\dot{v}_2^2 + B\dot{v}_1^2) + \left(3B_1 - \frac{M\omega^2}{2}\right)(x'^2 + y'^2) \\ + 3B_1L(v_1x' + v_2y') + \left(B_1L^2 + \frac{C-B}{2}\omega^2\right)v_2^2 \\ + \left(B_1L^2 + \frac{C-A}{2}\omega^2\right)v_1^2 = \text{constant} \dots \dots \dots (16) \end{aligned}$$

In the expression on the left the terms which involve the coordinates are not positive for all values of  $\omega$ . But whenever they are the same argument as was applied to (15) furnishes an absolute proof of stability. The necessary and sufficient conditions are that the terms in  $x'$  and  $v_1$  should be positive, and also those in  $y'$  and  $v_2$ . The second of these conditions is

$$(6B_1 - M\omega^2)\{2B_1L^2 + (C-B)\omega^2\} - 9B_1^2L^2 > 0, \quad \dots (17)$$

i.e., that  $\omega$  is less than the "whirling" speed with the symmetrical pulley of moments  $B$ ,  $B$ ,  $C$ ; and the first is similarly that  $\omega$  is less than that for the pulley of moments  $A$ ,  $A$ ,  $C$ . If  $\omega$  is so limited, then, the motion is certainly stable.

And the "speed" equation has shown that, if  $\omega$  exceeds such a limit, the motion is certainly unstable.

Equation (16) holds in the case of symmetry also. It is another integral of the equations of motion, independent of (15). But it does not serve to prove, as (15) does, that the motion is stable at all speeds.

A similar proof of absolute stability at all speeds in the case of elastic yielding discussed by Lees is easily obtained from his equations (34).

In the R. I. A. paper referred to I took as an instance of the weakness of the orthodox argument for stability an unsymmetrical top, properly chosen, and examined it in some detail. I now add that, in that case, we may easily obtain an equation of a type differing from (15) and (16) in having as the left-hand side the *difference*, instead of the sum, of two quadratic functions, each essentially positive, one involving the velocities and the other the coordinates.

LXVIII. *On the Theory of the Characteristic Curve of a Photographic Emulsion.*—Part II. By F. C. TOY, D.Sc., F.Inst.P.\*

[Communication No. 27 from the British Photographic Research Association Laboratory.]

**I**N a previous paper (Phil. Mag. 1922, xliv. pp. 352-371) it has been shown that for a fast emulsion the relation which is found to hold experimentally between the number of geometrically identical silver halide grains made developable and the light-intensity, is in qualitative agreement with the assumption that there exist in the grains specially light-sensitive points. These are distributed amongst the grains according to the laws of chance, and only grains which have at least one will be developable. The action of the light changes their condition in such a way that they become susceptible to the action of the developer and cause the reduction of the silver halide in their neighbourhood and hence ultimately of the whole grain. It is assumed that each nucleus does not necessarily require the same intensity to change it. The effect of variation of grain size is discussed, and it is shown that the experimental facts do not admit of the assumption regarding the similarity of the light-sensitive material in grains of various sizes in the case of a highly sensitive emulsion. The question of the validity of this assumption under any circumstances is obviously of great importance for the theory of the photographic process, and the evidence existing in its favour must now be closely examined.

The assumption was made by Svedberg (Phot. Journ. 1922, lxii. pp. 186-192, 310-320) that "the small and the larger grains in one and the same emulsion are built up of the same kind of light-sensitive material—just as if they were fragments of different size from one homogeneous silver bromide crystal." He says (p. 317) that for this hypothesis to be true a relationship which must hold is that

$$\nu_1/a_1 = \nu_2/a_2 = \dots = c = \text{constant}, \quad . \quad . \quad (8)$$

where  $\nu_1, \nu_2 \dots$  are the average number of centres per grain (which we have previously denoted by  $N_0$ ),  $a_1, a_2 \dots$  are the areas of the surfaces of the spherical grains of the emulsion

\* Communicated by Prof. A. W. Porter, F.R.S.

used, and  $c$  the average number of centres per square centimetre of grain surface. Considering only two sizes of grains, we have from (8),

$$\nu_2/\nu_1 = a_2/a_1, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

*i. e.* the ratio of the average number of nuclei per grain for two sizes of grain is constant and equal to the ratio of the surface areas. *It is therefore independent of the exposure.*

Let us now examine the experimental evidence which is taken by Svedberg as supporting this hypothesis. The data given by him in support of the validity of equation (8) have been rearranged in Tables I. and II. so as to show to what extent they support equation (9). The first column in Table I. gives the average area of grain for three different sizes, and the second and third columns the average number of nuclei per grain occurring at two different intensities (I). In Table II. the ratios of these numbers are calculated.

TABLE I.

Average area of grain.	I=1.00.	I=1.56.
$24.2 \times 10^{-9}$ cm. <sup>2</sup> .....	$\nu_1 = 0.48$	$\nu_1 = 0.94$
$46.5 \times 10^{-9}$ cm. <sup>2</sup> .....	$\nu_2 = 0.82$	$\nu_2 = 1.90$
$78.1 \times 10^{-9}$ cm. <sup>2</sup> .....	$\nu_3 = 1.47$	$\nu_3 = 2.59$

TABLE II.

I=1.00.	I=1.56.
$\nu_2/\nu_1 = 1.71$	$\nu_2/\nu_1 = 2.02$
$\nu_3/\nu_1 = 3.06$	$\nu_3/\nu_1 = 2.76$
$\nu_3/\nu_2 = 1.79$	$\nu_3/\nu_2 = 1.36$

Whether these figures may be taken as supporting the hypothesis is certainly open to doubt, for the following reasons:—

(1) The value of  $\nu_2/\nu_1$ ... etc., actually varies considerably with the exposure, the difference between the pairs of values given being of the order of 18, 10, and 24 per cent.

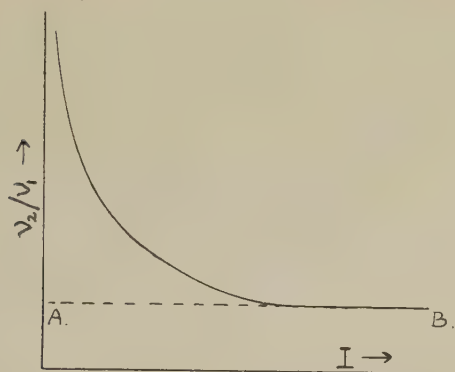
(2) These are values obtained only over a small range of intensities. The characteristic curve of the emulsion used extends over a range of intensities of about 1 to 5, so that these results are confined to a small part of the curve. Thus the experimental results seem to be very slender evidence in support of the hypothesis.

It was predicted in Part I. that the ratio  $\nu_2/\nu_1$  will not be constant, but will vary with the intensity. When this is small the ratio will be very large and may become infinite, and as the intensity is increased  $\nu_2/\nu_1$  decreases, at first rapidly, and ultimately approaches a constant value as more and more of the nuclei are affected. If, therefore, we plot  $\nu_2/\nu_1$  against the intensity we expect to obtain the kind of



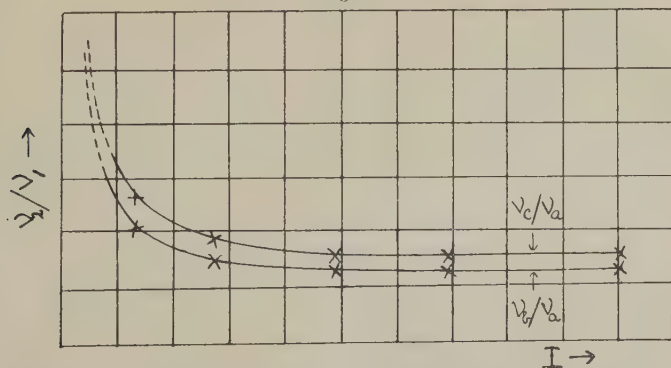
curve in fig. 1. On Svedberg's assumption we should simply get a straight line such as A B parallel to the I axis.

Fig. 1.



In the case of high speed emulsions it has already been shown that  $v_2/v_1$  is not independent of the exposure but, at the lower intensities, varies enormously with it. This is shown in fig. 2, where the curves are obtained from some of the data published in Part I. (p. 365). The similarity with the theoretical curve in fig. 1 is at once apparent.

Fig. 2.



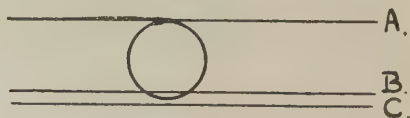
As has been shown in the first paper, this variation of the number of nuclei with the exposure is easily explained if we assume that the reason why large grains are more sensitive than small ones is not only that there exist more nuclei in the larger grains, but also that the average sensitivity of the nuclei increases with the grain size. Thus in the case of high-speed emulsions the different grains can by no means

be looked upon as "fragments of different size from one homogeneous silver bromide crystal"; rather must we regard the sensitive material as having, on the average, the same characteristics only for classes of equal-sized grains, and varying enormously from one class to another. This result is probably bound up with the ripening process.

If Svedberg's hypothesis that grains of all sizes contain the same kind of light-sensitive material were true for slow, spherical grains, then there must be some fundamental difference in the process of formation of the nuclei in slow and fast emulsions. It was, therefore, considered very important to test this hypothesis for a process emulsion over the whole range of its characteristic curve.

It was not possible to work with the same emulsion as Svedberg used, which was specially prepared by Messrs. Ilford Ltd. The nearest which could be obtained was the Ilford Process Emulsion, which is, however, the same type, the main difference apparently being that in the special emulsion the grain size is more uniform than in the ordinary process plates. Special "single layer" plates were made up in the usual way, and the exposure was made behind a step-wedge to white light. The plate was then developed to completion, washed, and dried without fixing. The method of measurement was practically the same as described in the previous paper. Two sizes of grain were selected for the experiment, one size (class 1) having a diameter of  $(0.60 \pm 0.05)\mu$ , and the other (class 2) a diameter of  $(0.75 \pm 0.05)\mu$ . Photo-micrographs of the emulsion were taken, over the whole range of the characteristic curve, on Barnet ortho process plates at a magnification of 2000, using a 2 mm. objective, a  $\times 14$  eyepiece, monochromatic green light, and N.A. 1.4. The negatives were then optically

Fig. 3.



projected on to a screen at a magnification of 5. The grains counted in class 1 were those whose images just fitted between parallel lines drawn on a white card, as in fig. 3, in such a way that  $AB < d < AC$ , where  $d$  is the diameter of the grain, and  $AB$  and  $AC$  are equal to 5.5 mm. and 6.5 mm. respectively.

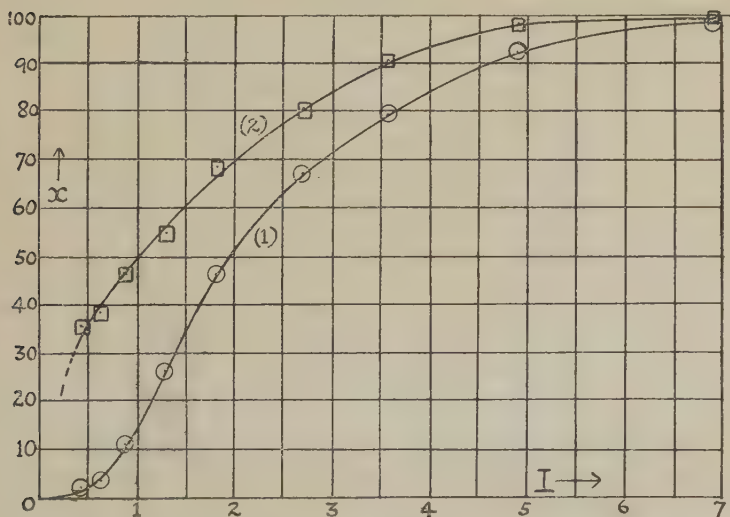
The method of counting previously described was modified

so as to eliminate any error due to fog and unevenness of coating. Let the total number of all sizes of grains, developed and undeveloped, in an arbitrary area of the unexposed portion of the plate be  $x_0$ , and the number of undeveloped grains of class 1 be  $y_0$ . Then the ratio of the number of undeveloped grains of class 1 to the total number of grains for the fogged portion of the plate is  $y_0/x_0$ , and this is independent of variations in the number of grains per unit area due to an uneven coating. In a certain area of the exposed part of the plate (which need not be the same as the area in the fogged portion) let the total number of grains of all sizes, developed and undeveloped, be  $x_e$ , and the total undeveloped of class 1 be  $y_e$ . Then, if this part of the emulsion had not been exposed, the number of grains of class 1 would have been  $y_0 x_e / x_0$ . Thus the number of class 1 which have been changed by the light-action is

$$y_0 x_e / x_0 - y_e,$$

out of a total of  $y_0 x_e / x_0$  unaffected by fog.

Fig. 4.



Therefore the percentage changed ( $x$ ) due to the light-action only is given by

$$x = 100 [1 - y_e x_0 / y_0 \cdot x_e]. \quad (10)$$

The values of  $x$  ( $x_1$  and  $x_2$ ) obtained from equation (10) for the two classes of grain are plotted in fig. 4, and the smooth curve values, together with the corresponding values

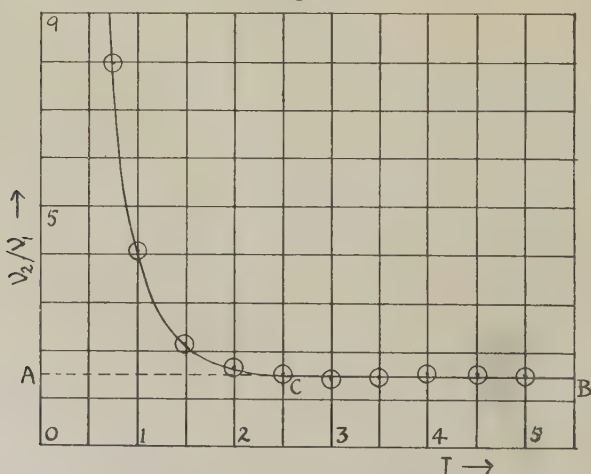
of  $\nu_1$  and  $\nu_2$  and the ratio  $\nu_2/\nu_1$ , are given in Table III. The values of  $\nu$  are calculated from the relation (found in Part I.),  $\log [100/(100-x)] = \nu$ .

TABLE III.

I.	$x_1$ .	$x_2$ .	$\nu_1$ .	$\nu_2$ .	$\nu_2/\nu_1$ .
0.500	93.5	98.5	2.734	4.200	1.54
0.450	89.8	97.0	2.271	3.507	1.54
0.400	84.5	94.6	1.865	2.917	1.56
0.350	79.0	90.0	1.561	2.303	1.48
0.300	73.0	84.2	1.309	1.843	1.41
0.250	63.5	78.0	1.009	1.514	1.51
0.200	52.0	70.5	0.734	1.221	1.67
0.150	35.5	61.5	0.438	0.956	2.18
0.100	16.0	50.8	0.174	0.709	4.07
0.075	7.0	44.0	0.073	0.580	7.96
0.050	2.0	35.5	0.020	0.438	21.90

The values of  $\nu_2/\nu_1$  are plotted against the intensity in fig. 5, and it will be seen that the curve is of exactly the

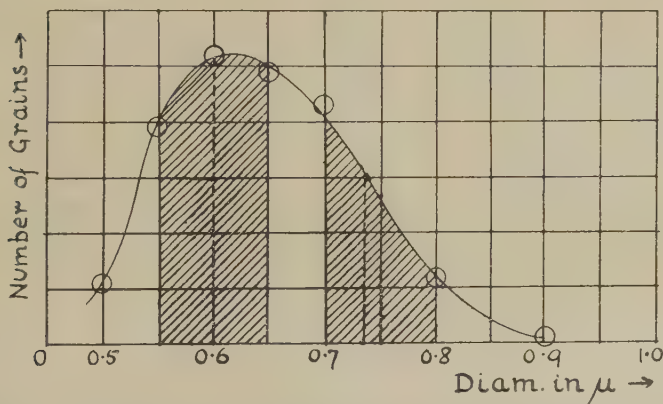
Fig. 5.



same form as in the case of a fast emulsion. Now Svedberg has proved that for a process emulsion with spherical grains all the nuclei are on the surface, so that *when all the nuclei are operative* (i. e. when  $I$  is large) we should expect the constant value to which  $\nu_2/\nu_1$  approaches to be the ratio of the surfaces of the two classes of grain. A fairly accurate

value of this can be very easily found. We are not justified in taking this ratio as exactly  $(7.5)^2/(6.0)^2$  unless there are just as many grains between  $0.55-0.60\mu$  and  $0.70-0.75\mu$  as there are between  $0.60-0.65\mu$  and  $0.75-0.80\mu$  respectively. In fig. 6 are plotted as ordinates the relative numbers of grains having fixed diameters. We have to take as the average of the diameters those values such that the ordinates through them divide the areas under the curve between the limits  $0.55 < \mu < 0.65$  and  $0.70 \leq \mu < 0.80$  into two equal parts.

Fig. 6.



These are found with sufficient accuracy to be  $0.600\mu$  and  $0.735\mu$ . Thus the theoretical ratio which  $\nu_2/\nu_1$  approaches is

$$(7.35/6)^2 = 1.50,$$

and this is the value obtained experimentally as near as it can be determined from the curve in fig. 5.

If the hypothesis of the similarity of the light-sensitive material in both classes of grain were true then the curve obtained in fig. 5 would be the straight line BCA. Thus it appears that the two intensities which Svedberg used were somewhere on the BC portion of the curve, which would explain the constancy, and the value, of the ratio  $\nu_2/\nu_1$  which he found.

If, therefore, equation (9) is a necessary consequence of Svedberg's hypothesis, we are led to conclude that

- (1) In any particular emulsion, whether fast or slow, the photosensitive material gives the same average response to light only in grains of the same size.
- (2) This response increases continuously with the grain size. As has been suggested, the explanation of this is probably intimately connected with the ripening process.



Svedberg has shown that in the case of spherical grains the nuclei which show up on partial development are, on the average, distributed equally over the grain surface; the author's results prove that for flat grains they occur more on the edges than anywhere else. To explain this it might be considered natural to attribute it to the existence of protuberances and sharp edges at which, the solution pressure being higher, the reaction will be facilitated. On this supposition the nuclei are not specially sensitive points at all, but are simply a result of the attack of the developer on the grain. This possibility has been mentioned in Part I. (p. 370), but the argument against it was not fully dealt with there. Obviously it is exceedingly important to settle this question, because if it were true that the nuclei observed are only due to the developer, then the whole argument in favour of specially sensitive points fails. The argument is based on the manner in which the average number of nuclei per grain varies with the intensity.

At a certain exposure the actual numbers of nuclei which were found on the members of a group of 50 equal-sized triangular flat grains were as follows:—

12112241100241112132041111

1200211123103201322232121.

The total number of nuclei is 77, made up of three 4's, five 3's, fifteen 2's, twenty 1's, and seven zeros. The average per grain, *counting every grain*, is  $77/50 = 1.54$ , which has already been denoted by  $\nu$ . The seven zeros represent the seven grains which are undevelopable because they have not got a nucleus. The average for 43 developable grains, which we will call  $\nu_d$ , is  $77/43 = 1.79$ . Now, if the nuclei which are found on partial development of developable grains are the result of the action of the developer upon specific points determined solely by the geometrical formation of the crystals, then  $\nu_d$  should be constant for all exposures for a given geometrical character of the crystal. The only reason why  $\nu$  decreases with a decrease of exposure would, on this assumption, be that there is an increase in the percentage of zeros. Starting with the experimental values given above, the kind of results which would be obtained is illustrated by the fictitious set of figures in Table IV. arranged in order of decreasing exposure.

TABLE IV.

Distribution of Nuclei.	$\nu$ .	$\nu_d$ .
1211224110024112132041111 } .....	1.54	1.79
1200211123103201322232121 }		
0124300121100011111200200 } .....	1.08	1.80
0342220020001122020200113 }		
0001002000400100100011300 } .....	0.72	1.80
0011010000300012202212022 }		
0000001000200001030000210 } .....	0.36	1.80
0000000202000000031000000 }		

Here  $\nu_d$  is constant and  $\nu$  decreases with the exposure only because the percentage of zeros increases.

On the other hand, if the theory of "preference" points is true, then, as the exposure decreases,  $\nu_d$  should also decrease down to a limiting value of unity. Now a set of figures which was actually obtained in an experiment is given in Table V., and it will be seen to be exactly as predicted by this theory. Such a result can be repeated any number of times, and would be practically impossible if the nuclei were due only to the action of the developer.

TABLE V.

Distribution of Nuclei.	$\nu$ .	$\nu_d$ .
1211224110024112132041111 } .....	1.54	1.79
1200211123103201322232121 }		
2201111111311210003112220 } .....	1.24	1.51
2101110112112012103311222 }		
0021121110000110121100001 } .....	0.76	1.31
1220110200011020021110210 }		
0110010010001001000101000 } .....	0.42	1.17
1000012002000100011101002 }		

There remains, as a serious alternative to the theory of specially sensitive points, the theory, put forward by Silberstein, and based on the idea of the bombardment of the grains by discrete light quanta (Phil. Mag. 1922, xlv. pp. 257, 956). Experimental results which are brought forward as supporting the theory are given by Trivelli and Righter (Phil. Mag. 1922, xlv. p. 252). The interpretation of these results has already been briefly criticized by the author (B. J. P. 1922, lxix. pp. 443-444). It is proposed now to summarize, as concisely as possible, the evidence for and against this light-quantum theory.

The fundamental equation, which Silberstein says is of

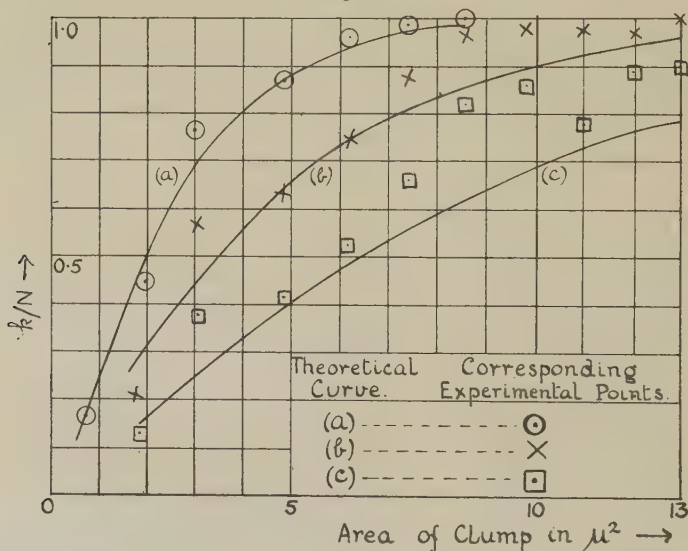
more than sufficient mathematical accuracy under the conditions prevailing in all practicable experimental cases, is

$$k/N = 1 - e^{-na}, \quad . . . . . (11)$$

where  $a$  is the projected area of the grain,  $k$  the number of grains hit,  $N$  the total number of grains per unit area, and  $n$  the number of light-quanta impinging upon unit area.

Now, Trivelli and Righter have found that in *certain cases* where the grains of an emulsion are grouped together in clumps the whole of the clump acts as a single grain for development to the limit, *i. e.*, the sufficient condition for the whole clump to be reduced is that one grain in it has been changed by the light. Admittedly this fact is proved by Trivelli's recent communication (B. J. P. 1922, lxix. p. 687).

Fig. 7.



Having proved this for a particular emulsion they proceed, with it, to test equation (11) for *clumps* of grains, rather than for single grains, on the ground that this will provide a much more severe test because a much bigger variation in  $a$  can thus be obtained. The agreement between the theoretical and observed values for three cases which are given, is shown in fig. 7, and, at all events as regards cases (b) and (c), it is difficult to take it as proved that the experimental results are in accordance with equation (11) in the case of clumps. Especially is this so in view of the fact

that the only point of agreement—viz., that the average experimental value of  $k/N$  does certainly increase with  $a$  as does the theoretical value—is what is to be expected on the theory of specially sensitive points.

When we come to test the equation in the simplest possible cases, we find that it breaks down completely. For example, it will not fit, even approximately, the curves given in this paper or in Part I. for sets of equal-sized grains. Again, if we consider the case of two sets of grains of different sizes we have:—

$$\log (N/(N-k_1)) = na_1 \quad \text{and} \quad \log (N/(N-k_2)) = na_2.$$

The left-hand side of the equation has already been shown to be equal to  $\nu$ , so that

$$\nu_1 = na_1 \quad \text{and} \quad \nu_2 = na_2,$$

$$\text{or} \quad \nu_2/\nu_1 = a_2/a_1 = \text{constant},$$

which is the same condition as required by Svedberg's hypothesis, and which has been shown to be untenable.

In discussing the dependence of the photographic effect on the wave-length, Silberstein deduces the equation (p. 266)

$$s = \beta E \cdot a\lambda, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where  $E$  is the incident light-energy (exposure),  $\lambda$  the wave-length of the monochromatic light,  $\beta$  a constant, and  $s=na$ .

But, from equation (11),

$$na = \log [N/(N-k)] = A.$$

$$\therefore A = \beta E \cdot a\lambda.$$

Thus for a constant value of  $a$  and  $E$ , we have

$$A \propto \lambda,$$

i. e.,  $A$  would "increase steadily with the wave-length of the incident light up to the photoelectric critical value  $\lambda_c$  and then drop suddenly to zero." Silberstein goes on to say that this abrupt drop to zero might only occur in a set of equal-sized grains, since grains of different sizes may have different values of  $\lambda_c$ . "For then, although the curve of each grain class would end abruptly, the superposition of such curves ending over a range of different abscissæ might properly displace and smooth out the resultant maximum." This question of maximum and the manner in which the  $A, \lambda$  curve drops to zero need not be considered

here, since the maximum cannot be reached until the wave-length is long enough to include some of the critical frequencies. In the author's experiments now to be described, and which have not yet been published, this state was never reached, so that according to Silberstein  $A$  should vary strictly as  $\lambda$ . The experimental method was as follows:—Sections of a single layer Ilford process emulsion were exposed for equal times to the same incident intensity (as measured by a thermopile) of different monochromatic lights. The source was the mercury arc, and the wave-lengths used were  $\lambda_1=436$  and  $\lambda_2=406$  (mean of 405 and 408). Therefore the ratio of  $A_1/A_2$  ought to have been about  $44/41=1.07$ . The ratios found in three different cases were 0.40, 0.36, and 0.37. Therefore instead of becoming greater,  $A$  decreases very rapidly as the wave-length increases.

All these experiments go to show that a relation of the type of equation (11) is fundamentally wrong as expressing the true mechanism of photographic exposure. This is what Slade and Higson concluded (Proc. Roy. Soc. 1920, A, xeviii. p. 154) over three years ago.

In conclusion, the author wishes to express his thanks to Dr. Slater Price (Director of Research of the British Photographic Research Association) and to Professor Porter for their interest in this work.

LXIX. *The Gravitational Field of a Particle on Einstein's Theory.* By J. T. COMBRIDGE, B.A., University of London, King's College\*.

§ 1. **T**HE most general form for the line-element in a static gravitational field symmetrical about the origin is

$$ds^2 = -e^\lambda dr^2 - e^\mu (r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + e^\nu dt^2, \quad (1.0)$$

where  $\lambda$ ,  $\mu$ , and  $\nu$  are functions of  $r$  only.

It is well known that the equations of the gravitational field are insufficient to determine  $\lambda$ ,  $\mu$ , and  $\nu$ , and the general procedure has been to assume some relation between them, as, for example,  $\lambda=0$  †,  $\mu=0$  ‡,  $\lambda-\mu=0$  §,  $\lambda+2\mu+\nu=0$  ||.

This lack of determinacy is due to the fact that the

\* Communicated by Prof. G. B. Jeffery.

† J. Droste, Proc. Amsterdam Acad. vol. xix.

‡ Schwarzschild, *Sitzungsber. Ber.* 1916, Feb.

§ F. W. Hill and G. B. Jeffery, Phil. Mag. Ser. 6, vol. xli., and an approximate solution by de Sitter, Proc. Amst. Acad. vol. xix.

|| Einstein, *Sitzungsber. Ber.* 1915, Nov.



condition of symmetry about the origin imposes on the measurement of the coordinate  $r$  no other condition than that whatever be the mode of measurement in one radial direction the same mode must be adopted in every other radial direction. Subject to this condition, any function of  $r$  may be taken as the coordinate instead of  $r$ , the substitution of this function in (1.0) giving, of course, a different form for  $ds^2$ .

The purpose of this note is to show explicitly how this indeterminacy arises in the solution of the field equations, and to determine the  $\lambda$ ,  $\mu$ , and  $\nu$  of equation (1.0) in terms of a single arbitrary function. The notation adopted is that employed by Eddington in his "Report on the Relativity Theory of Gravitation."

§ 2. We take the line-element to be of the form

$$ds^2 = -e^\lambda dr^2 - e^\mu r^2 d\theta^2 - e^\mu r^2 \sin^2 \theta d\phi^2 + e^\nu dt^2, \quad (2.0)$$

where  $\lambda$ ,  $\mu$ ,  $\nu$  are functions of  $r$  only, and we try for a solution of

$$\begin{aligned} G_{\sigma\tau} \equiv \{\sigma\alpha, \beta\}\{\tau\beta, \alpha\} - \frac{\partial}{\partial x_\alpha}\{\sigma\tau, \alpha\} - \{\sigma\tau, \alpha\} \frac{\partial}{\partial x_\alpha} (\log \sqrt{-g}) \\ + \frac{\partial^2}{\partial x_\sigma \partial x_\tau} (\log \sqrt{-g}) = 0. \end{aligned}$$

From (2.0) we have

$$\begin{aligned} g_{11} &= -e^\lambda, & g_{33} &= -r^2 \sin^2 \theta e^\mu, \\ g_{22} &= -r^2 e^\mu, & g_{44} &= e^\nu, \\ g_{\sigma\tau} &= 0 \text{ if } \sigma \neq \tau, \\ g &= -r^4 \sin^2 \theta e^{\lambda+2\mu+\nu}. \end{aligned}$$

The Christoffel Symbols which do not vanish are :

$$\begin{aligned} \{11, 1\} &= \frac{1}{2}\lambda', \\ \{12, 2\} &= \frac{1}{2r}(\mu'r + 2), & \{22, 1\} &= -\frac{1}{2}r(\mu'r + 2)e^{\mu-\lambda}, \\ \{13, 3\} &= \frac{1}{2r}(\mu'r + 2), & \{33, 1\} &= -\frac{1}{2}r(\mu'r + 2)\sin^2 \theta e^{\mu-\lambda}, \\ \{14, 4\} &= \frac{1}{2}\nu', & \{44, 1\} &= \frac{1}{2}\nu'e^{\nu-\lambda}, \\ \{23, 3\} &= \cot \theta, & \{33, 2\} &= -\sin \theta \cos \theta, \end{aligned}$$

where dashes denote differentiation with respect to  $r$ , as, indeed, they will throughout.

Omitting terms which vanish, and taking account of the fact that, although its terms do not all vanish separately, yet

$$G_{12} \equiv 0 \quad \text{identically,} \quad \text{and that}$$

$$G_{33} \equiv \sin^2 \theta G_{22},$$

we get three equations, from  $G_{11}=0$ ,  $G_{22}=0$ ,  $G_{44}=0$ , respectively. They are \* :—

$$\mu'' + \frac{1}{2}\nu'' + \frac{1}{2}\mu'^2 + \frac{1}{4}\nu'^2 - \frac{1}{2}\lambda'\mu' - \frac{1}{4}\lambda'\nu' - \frac{1}{r}\lambda' + \frac{2}{r}\mu' = 0,$$

$$\mu'' + \mu'^2 - \frac{1}{2}\lambda'\mu' + \frac{1}{2}\mu'\nu' - \frac{1}{r}\lambda' + \frac{4}{r}\mu' + \frac{1}{r}\nu' = \frac{2}{r^2}(e^{\lambda-\mu} - 1),$$

$$-\frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 - \frac{1}{2}\mu'\nu' + \frac{1}{4}\lambda'\nu' - \frac{1}{r}\nu' = 0.$$

These can be combined to give

$$\mu'' - \frac{1}{2}\lambda'\mu' + \frac{1}{2}\mu'^2 - \frac{1}{2}\mu'\nu' - \frac{1}{r}\lambda' + \frac{2}{r}\mu' - \frac{1}{r}\nu' = 0,$$

$$\nu'' + \frac{1}{2}\nu'^2 + \mu'\nu' - \frac{1}{2}\lambda'\nu' = 0,$$

$$\frac{1}{2}\mu'^2 + \mu'\nu' + \frac{2}{r}\mu' + \frac{2}{r}\nu' + \frac{2}{r^2} = \frac{2}{r^2}e^{\lambda-\mu},$$

which may be reduced to

$$\frac{\partial}{\partial r} \left\{ r e^{-\frac{1}{2}(\lambda-\mu+\nu)} \frac{\partial \mu}{\partial r} \right\} + 2 \frac{\partial}{\partial r} \left\{ e^{-\frac{1}{2}(\lambda-\mu+\nu)} \right\} = 0, \quad (2.1)$$

$$\frac{\partial}{\partial r} \left\{ r^2 e^{-\frac{1}{2}(\lambda-2\mu-\nu)} \frac{\partial \nu}{\partial r} \right\} = 0, \quad (2.2)$$

$$(\mu'r + 2)(\mu'r + 2 + 2\nu'r) = 4e^{\lambda-\mu}. \quad (2.3)$$

§ 3. Integrating (2.1) and (2.2) we have

$$\mu'r + 2 = b e^{\frac{1}{2}(\lambda-\mu+\nu)}, \quad (3.11)$$

$$\nu'r^2 = a e^{\frac{1}{2}(\lambda-2\mu-\nu)}, \quad (3.12)$$

where  $a$  and  $b$  are constants of integration.

Now (3.11) may be written

$$\frac{\partial}{\partial r} (r e^{\frac{1}{2}\mu}) = \frac{1}{2} b e^{\frac{1}{2}(\lambda+\nu)},$$

from which it may be seen that, if we make the boundary

\* Compare de Sitter, Monthly Notices R. A. S. vol. lxxvi. p. 712.

conditions at infinity such that  $\lambda$ ,  $\mu$  and  $\nu \rightarrow 0$  as  $r \rightarrow \infty$ , then we must have  $b=2$ .

In (3.11) and (3.12) make the substitution

$$re^{\frac{1}{2}\mu} = f(r) \equiv f.$$

Then

$$e^{\frac{1}{2}(\lambda+\nu)} = f', \quad . \quad . \quad . \quad . \quad . \quad (3.21)$$

$$e^{\frac{1}{2}(\lambda-\nu)} = \nu' f^2 / a. \quad . \quad . \quad . \quad . \quad (3.22)$$

Eliminating  $e^\lambda$ ,

$$\nu' e^\nu = a f' / f^2,$$

so that

$$e^\nu = 1 - \frac{a}{f}, \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

the constant of integration having been adjusted so that  $\nu \rightarrow 0$  as  $r \rightarrow \infty$ .

Putting this value of  $e^\nu$  in (3.21) we find that

$$e^\lambda = \frac{f'^2}{1 - a/f} \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

with

$$e^\mu = \frac{f^2}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

Thus from equations (2.1) and (2.2) we have  $\lambda$ ,  $\mu$ , and  $\nu$  in terms of a single function  $f$ . On substituting in (2.3) we find that, instead of giving an equation to find  $f$ , (2.3) is satisfied identically, showing that  $f(r)$  is an *arbitrary* function of  $r$ .

Eliminating  $f$  between (3.3) and (3.5) we have

$$e^\nu = 1 - \frac{a}{r} e^{-\frac{1}{2}\mu},$$

and we conclude that the equations

$$G_{\sigma r} = 0$$

will be satisfied so long as this relation between  $\mu$  and  $\nu$  is satisfied.

§ 4. Our line-element is therefore of the form

$$ds^2 = -e^\lambda dr^2 - f^2(d\theta^2 + \sin^2\theta d\phi^2) + e^\nu dt^2, \quad . \quad (4.0)$$

where  $f$  is an arbitrary function of  $r$ , and  $e^\lambda$  and  $e^\nu$  are functions of  $f$  which are given explicitly in equations (3.4) and (3.3) but have been retained here for brevity. This form

was derived by Droste and Schwarzschild from the equation

$$\delta \iiint G \sqrt{-g} dx_1 dx_2 dx_3 dx_4 = 0.$$

The Christoffel Symbols which do not vanish are now:—

$$\begin{aligned} \{11, 1\} &= \frac{1}{2}\lambda', \\ \{12, 2\} &= f'/f & \{22, 1\} &= (f-a)/f', \\ \{13, 3\} &= f'/f & \{33, 1\} &= \sin^2 \theta (f-a)/f', \\ \{14, 4\} &= \frac{1}{2}\nu' & \{44, 1\} &= \frac{1}{2}\nu' e^{\nu-\lambda}, \\ \{23, 3\} &= \cot \theta & \{33, 2\} &= -\sin \theta \cos \theta. \end{aligned}$$

The equations of motion of a particle moving in this field are

$$\frac{\partial^2 x_\sigma}{\partial s^2} + \{\alpha\beta, \sigma\} \frac{\partial x_\alpha}{\partial s} \frac{\partial x_\beta}{\partial s} = 0;$$

that is,

$$\begin{aligned} \frac{d^2 r}{ds^2} + \frac{1}{2}\lambda' \left(\frac{dr}{ds}\right)^2 + \frac{f-a}{f'} \left(\frac{d\theta}{ds}\right)^2 + \frac{f-a}{f'} \sin^2 \theta \left(\frac{d\phi}{ds}\right)^2 \\ + \frac{1}{2}\nu' e^{\nu-\lambda} \left(\frac{dt}{ds}\right)^2 = 0, \quad (4.1) \end{aligned}$$

$$\frac{d^2 \theta}{ds^2} + \frac{2f'}{f} \frac{dr}{ds} \frac{d\theta}{ds} - \sin \theta \cos \theta \left(\frac{d\phi}{ds}\right)^2 = 0, \quad (4.2)$$

$$\frac{d^2 \phi}{ds^2} + \frac{2f'}{f} \frac{dr}{ds} \frac{d\phi}{ds} + 2 \cot \theta \frac{d\theta}{ds} \frac{d\phi}{ds} = 0, \quad (4.3)$$

$$\frac{d^2 t}{ds^2} + \nu' \frac{dr}{ds} \frac{dt}{ds} = 0. \quad (4.4)$$

If initially the particle moves in the plane  $\theta = \frac{\pi}{2}$  so that initially  $\frac{d\theta}{ds} = 0$ , then, from (4.2),

$$\frac{d^2 \theta}{ds^2} = 0,$$

and so  $\frac{d\theta}{ds} = 0$  always. Hence (4.3) and (4.4) simplify, and integrate to give

$$f^2 \frac{d\phi}{ds} = \text{const.} = h, \quad (4.5)$$

$$e^\nu \frac{dt}{ds} = \text{const.} = k. \quad (4.6)$$

Instead of using (4.1) with these we use (4.0), which reduces to

$$\left(\frac{dr}{ds}\right)^2 + \frac{f(f-a)}{f'^2} \left(\frac{d\phi}{ds}\right)^2 = \frac{k^2}{f'^2} - \frac{f-a}{ff'^2},$$

$$\text{i. e.} \quad f'^2 \left(\frac{dr}{ds}\right)^2 + f^2 \left(\frac{d\phi}{ds}\right)^2 = k^2 - \frac{f-a}{f} + af \left(\frac{d\phi}{ds}\right)^2,$$

$$\text{i. e.} \quad \left(\frac{df}{ds}\right)^2 + f^2 \left(\frac{d\phi}{ds}\right)^2 = k^2 - 1 + \frac{a}{f} + \frac{ah^2}{f^3}.$$

§ 5. It is interesting to note how solutions previously found by making *a priori* assumptions as to the indeterminacy of the line-element may be obtained from the results of § 3.

When we take  $\mu=0$ , then  $f=r$ , and, writing  $2m$  instead of  $a$ , the constant of integration, we have from (3.3) and (3.4),

$$e^\nu = 1 - \frac{2m}{r},$$

$$e^\lambda = e^{-\nu}.$$

When we take  $\lambda=\mu$  we have, from (3.4) and (3.5), a differential equation in  $f$ :—

$$r^2 f'^2 = f(f-a),$$

the solution of which is

$$f = \frac{a(r+a_1)^2}{4a_1r}.$$

If we determine  $a_1$ , the constant of integration, by noting that

$$e^{\frac{2}{3}\mu} = \frac{f}{r} = \frac{a}{4a_1} \left(1 + \frac{a_1}{r}\right)^2,$$

and by the condition that  $\mu \rightarrow 0$  as  $r \rightarrow \infty$ , we find that

$$4a_1 = a,$$

$$\frac{f}{r} = \left(1 + \frac{a}{4r}\right)^2.$$

Writing again  $2m$  instead of  $a$  we get

$$e^\lambda = e^\mu = \left(1 + \frac{m}{2r}\right)^4,$$

$$e^\nu = \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2.$$



The condition  $\lambda + 2\mu + \nu = 0$  is seen to be equivalent to the condition  $\mu = 0$ , for, from (3.3), (3.4), and (3.5),

$$e^{\lambda+2\mu+\nu} = \frac{f^4 f'^2}{r^4},$$

so that  $\lambda + 2\mu + \nu = 0$  means

$$f^4 f'^2 = r^4,$$

whence

$$f = \pm r,$$

and since  $r^2 e^\mu = f^2$ , the result follows.

## LXX. On the Inversion of Bernoulli's Theorem.

By CHARLES JORDAN\*.

1. **S**UPPOSING that a certain phenomenon could have taken place in one, and only one, of several different ways; let us denote by  $\omega_i$  the probability *a priori* (*i. e.* before the phenomenon has occurred) that the  $i$ th way is followed, and by  $P_i$  the probability that in this case the phenomenon occurs. The probability *a posteriori*  $\Delta W_i$  (*i. e.* after the event) that the phenomenon has been obtained in the  $i$ th way is given by Bayes' Theorem:

$$\Delta W_i = \frac{\omega_i P_i}{\sum \omega_i P_i} \cdot \cdot \cdot \cdot \cdot \quad (1)$$

If  $\omega_i$  and  $P_i$  are given quantities, no objection at all can be made to this formula.

Often  $P_i$  is given while  $\omega_i$  is unknown; then we must adopt some hypothesis concerning the values of these later probabilities. This will not prevent us from using the theorem, if we remember that in each case of applied probability we are obliged to accept a similar hypothesis concerning the equally probable cases; as, in spite of Kriess and other philosophers, there is no objective criterion for them. In both cases, if the obtained results conform, more or less, to those given by experience, our choice of the hypothesis will be more or less justified. No hypothesis can be proved by experience to be wrong in general, but only in particular cases. For example, the often debated Rule of Succession based on the hypothesis  $\omega_i = \text{const.}$ , can in certain

\* Communicated by the Author.

cases be proved conformable to the results of experiment, in others adverse. Thus a contrary hypothesis,  $\omega_i = \binom{n}{i} (\frac{1}{2})^n$ , given by Boole and also by Stumpf, can be proved right or wrong, according to the case selected.

In this paper, an application of Bayes' theorem, for the inversion of Bernoulli's formula, will be made; though this use has often, quite unjustifiably, been contested. The fact that this inversion is generally stated in the textbooks in a very unsuitable way, has contributed to bring it into bad repute.

2. *Bernoulli's Theorem.* The probability  $p$  of the single event being given, the probability that among  $n$  observations there be  $\nu$  favourable ones is expressed by

$$P_i = \binom{n}{\nu} p^\nu q^{n-\nu}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where  $q = 1 - p$ .

Let us denote the ratio  $\nu/n$  by  $\bar{p}$ ; and let

$$\bar{p} - p = \frac{x}{n} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting for  $\nu$  in the formula (2) its value taken from (3)  $\nu = x + np$ , we have

$$P(x) = \binom{n}{x+np} p^{np+x} q^{nq-x} \quad . \quad . \quad . \quad . \quad (4)$$

For large values of  $n$ , this probability is given by the asymptotic formula

$$P(x) \sim \frac{k}{\sqrt{\pi}} e^{-k^2 x^2}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where  $k = 1/\sqrt{2pqn}$ .

If  $p$  is known (4) and (5) express the probability that  $p$  satisfies (3).

Let us put in (5)  $kx = t$ , as according to (3) we have  $\Delta x = 1$ ; therefore  $\Delta t = k$ ; and it follows that

$$\Theta(\lambda k) = \frac{2}{\sqrt{\pi}} \int_0^{\lambda k} e^{-t^2} dt \quad . \quad . \quad . \quad . \quad (6)$$

is the probability that  $\bar{p}$  satisfies the inequality (7),

$$|\bar{p} - p| < \frac{\lambda}{n} \quad . \quad . \quad . \quad . \quad (7)$$

3. *Inverse Problem.* The event having occurred  $\nu$  times out of  $n$ , hence  $\bar{p}$  is known; required the probability  $\Delta W_i$  that the probability  $p = p_i$  of the single event satisfies (3). If we accept an hypothesis concerning the *a priori* probabilities  $\omega_i$  of  $p_i$ , the problem is solved by Bayes' theorem.

Let us admit that  $p_i$  may take indifferently one of the values  $0, \frac{1}{m}, \frac{2}{m}, \dots, 1$ . Therefore the probability *a priori* of  $p_i = \frac{i}{m}$  will be  $\omega_i = \frac{1}{m+1}$ . And according to Bayes' theorem,

$$\Delta W_i = \frac{\frac{1}{m+1} \binom{n}{\nu} \left(\frac{i}{m}\right)^\nu \left(\frac{m-i}{m}\right)^{n-\nu}}{\sum \frac{1}{m+1} \binom{n}{\nu} \left(\frac{i}{m}\right)^\nu \left(\frac{m-i}{m}\right)^{n-\nu}}.$$

If  $S$  is the denominator of this expression, it is easy to show that

$$\lim_{m \rightarrow \infty} S = \frac{1}{n+1}.$$

$\Delta W_i$  will moreover satisfy (3) if

$$\bar{p} - \frac{i}{m} = \frac{x}{n}, \quad \dots \quad (8)$$

then the required probability will be

$$\Delta W_i = \frac{1}{S(m+1)} \binom{n}{\nu} \left(\bar{p} - \frac{x}{n}\right)^\nu \left(\bar{q} + \frac{x}{n}\right)^{n-\nu}, \quad \dots \quad (9)$$

where  $\bar{q} = 1 - \bar{p}$ .

And if  $m, n, \nu$ , and  $n - \nu$  are very large, the asymptotic value of this probability is

$$\Delta W_i = \frac{\bar{k}}{\sqrt{\pi}} e^{-\bar{k}^2 x^2} \Delta x, \quad \dots \quad (10)$$

where  $\bar{k} = 1/\sqrt{2\bar{p}q n}$ ;  $\Delta x$  is, according to (8), equal to  $-\frac{n}{m}$ .

Hence if  $p$  is known, (9) and (10) are expressions of the probability that  $p$  satisfies (3).

From (10) we conclude that if  $p$  is known the probability that  $p$  satisfies (7) is given by the same formula (6) as the solution of the corresponding direct problem, only  $\bar{k}$  must be substituted for  $k$ .

The inverted theorem presented in this manner seems more acceptable than when given in the usual way\*. If the interpreted problem is such that the hypothesis  $\omega_i = \text{const.}$  is justified, the results proceeding from the use of this formula will be confirmed by experience.

January 1923.

LXXI. *The Fluorescence and Coloration of Glass produced by  $\beta$ -rays.* By J. R. CLARKE, M.Sc., F.Inst.P., Assistant Lecturer in Physics, The University, Sheffield†.

AT the 1922 meeting of the British Association, Mr. J. Ewles described some investigations into the fluorescence produced in various substances by cathode rays. He concluded that the fluorescence was due to a change in the state of molecular aggregation of the substances. Some observations on the coloration of glass by  $\beta$ -rays point to the same conclusion. Three specimens of glass tubing (of approximate composition 1.5 Na<sub>2</sub>O, 0.5 CaO, 6 SiO<sub>2</sub>; 1.4 Na<sub>2</sub>O, 0.6 CaO, 6 SiO<sub>2</sub>; 1.2 Na<sub>2</sub>O, 0.8 CaO, 6 SiO<sub>2</sub>; referred to as glasses 6, 7, and 9 respectively) were obtained from the Glass Department of this University, and kept in radium emanation, the supply of emanation being renewed from time to time till they were coloured a deep brown and ceased to fluoresce. Further radiation did not alter the colour. The walls of the tubing were about 1.5 mm. thick, but previous experiments with rods had shown that the coloration was due to  $\beta$ -rays, extending only to a depth to which these rays would penetrate. When the specimens were completely fatigued, each was divided into four, and a piece from each placed in an oven maintained at 110° C. They immediately began to fluoresce, and the time during which fluorescence persisted was noted. At the end of this time the glasses were decolorized. The other pieces were similarly heated to 180°, 235°, and 350°. The following results were obtained:—

Temperature.	Time of fluorescence in minutes.		
	Glass 6.	Glass 7.	Glass 9.
110° .....	13	13	13
180° .....	3	3	3
235° .....	1.25	1.25	1.5
350° .....	0.5	0.5	0.75

\* See Czuber, 'Wahrscheinlichkeitsrechnung,' Third edition, 1914, vol. i. pp. 203-206.

† Communicated by the Author.

A piece of glass 9, also coloured, was heated to about  $95^{\circ}$ , and the fluorescence persisted for 15 hours. The intensity of the fluorescence was approximately inversely proportional to the time for which it persisted.

The conditions were such that the results are only approximate, but if curves be drawn showing the relation between the temperature and the time of fluorescence, they indicate, by extrapolation, that the glasses would be decolorized immediately if heated to a temperature between  $500^{\circ}$  and  $600^{\circ}$ , glass 9 requiring a slightly higher temperature than the others. The annealing temperatures of these glasses have been determined by Turner and English\*, and found to be  $526^{\circ}$ ,  $538^{\circ}$ , and  $562^{\circ}$  respectively. Tool and Valasek† have shown that there is an increased emission of heat when glass is cooled through the annealing temperature, indicating a change in the state of molecular aggregation at this temperature. Annealing proceeds at lower temperatures if the heating is sufficiently prolonged, and thus it is probable that if an unstable state of aggregation exists, it will slowly change to a stable state even when the temperature is low. The decoloration of the glasses, therefore, appears to be connected with a change in the state of molecular aggregation, coloration being the reverse process. As both coloration and decoloration were accompanied by fluorescence, it is probable that the fluorescence is due to this change.

It may be mentioned, further, that altogether nineteen different glasses have been radiated, fourteen of which were coloured brown and the remainder purple. I am indebted to my wife for chemical examination of these glasses. It was found that the five coloured purple all contained manganese, while the others did not. It has been stated frequently that soda-containing glasses are turned blue or purple by radiation. Varying amounts of soda were present in the glasses radiated, but the purple colour was not produced in the absence of manganese.

Sheffield,  
January 26th, 1923.

\* Journ. Soc. Glass Tech. iii. p. 125 (1919).

† Bureau of Standards Papers, No. 358 (1920).



LXXII. *Notes on X-ray Scattering and on J Radiations.*  
 By C. G. BARKLA, F.R.S., and RHODA R. C. SALE, M.A.,  
*University of Edinburgh* \*.

EARLY experiments by one of the writers on the transmission of Röntgen radiation through matter demonstrated the existence of secondary X-radiations of two and only two types—scattered X-radiation and fluorescent (characteristic) X-radiation. The former was not found to differ appreciably from the primary radiation exciting it, except in polarization, distribution, etc., the other was seen to be independent of the character (or wave-length) of the primary radiation.

The laws governing the emission of the scattered radiation were found to be of a very simple nature, and they led directly to the conclusion—a conclusion which has since been fully confirmed—that the scattering was produced by the electrons constituent in matter. One of these laws was that the intensity of scattering by *light* elements was independent of the penetrating power of the X-rays, *i. e.* the proportion of the energy of a beam lost by scattering during transmission through a thin layer of matter was independent (approximately) of the wave-length of the radiation used. Such a law, of course, had its limitations, and gave only the most important feature of the relationship—or lack of relationship—between scattering and quality of radiation. Whereas other properties of the rays varied enormously, the scattering was approximately constant during a variation of wave-length.

Later experiments† showed that when *heavy* elements were traversed by X-rays there was a considerable variation of scattering with wave-length, and that the heavier the element the greater in general was the rate of variation. It, of course, followed as highly probable that such variation existed when scattering was produced by light elements, though to a much less extent. It was also seen that the scattering by heavy elements approximated to equality with that by light elements mass for mass, when the wave-length was reduced. (It would probably be more accurate to say that the scattering per atom approximated to proportionality with the atomic number and not the atomic weight, though the experiments on heavy elements were not sufficiently extended to distinguish between the two.) The increase of scattering with the longer waves was explained as consequent

\* Communicated by the Authors.

† Barkla and Miss Dunlop, *Phil. Mag.* March 1916.

upon the superposition in phase of effects from neighbouring electrons when the wave-length became comparable with the distances of the electrons apart.

In describing the later results, the variation of scattering with wave-length was not given absolutely, but by comparison with that in aluminium, a substance in which the variation was seen to be comparatively small.

There were several reasons why a further investigation of this variation seemed desirable. Absolute instead of comparative measurement, and greater accuracy were needed to furnish both a further test of the theory and to provide possible information regarding atomic structure.

In addition, it was seen that exact experiments would give valuable information on the question of a possible change in penetrating power in the process of scattering, and regarding the emission of a characteristic X-radiation of shorter wave-length than that of series K by elements of low atomic weight—such as had been strongly suggested by the experimental results of Barkla and Miss White\*.

Regarding the former there has always appeared the possibility of a change in the properties of radiation produced by the process of scattering by electrons—particles of finite dimensions subject to restraining forces within the atom—or more probably still by groups of electrons where the relative dimensions are such as to make a group the scattering unit.

Though secondary X-radiations apparently differing considerably from the primary may very readily be obtained even when the lightest elements are used as scattering substances, we have never obtained convincing evidence of a change of quality of radiation *in the process of scattering*, except such as is given by polarization in the scattered radiation. One of our purposes was to look more closely for evidence of transformation—apart from the transformation (previously mentioned) occurring in the phenomenon of X-ray fluorescence, when characteristic radiations are emitted by the substance. The results of these preliminary researches are set down below. The investigation is being continued.

#### *Variation of Scattering with Wave-length.*

Barkla and Miss Dunlop† first showed the increase of scattering with wave-length in copper, silver, tin, and lead, by comparing the intensity of scattered radiation from

\* Many of these are still unpublished. See Barkla, Bakerian Lecture, Phil. Trans. Roy. Soc. 1917; Barkla and White, Phil. Mag. Oct. 1917.

† Phil. Mag. March 1916.

such substances with the scattering from aluminium. The results are reproduced in Table I. They were corrected for absorption etc., and show relative intensities from equal masses of these elements when exposed to the same radiation.

TABLE I.

Showing relative intensities of scattered X-radiation of various wave-lengths from several metals.

X-radiation.		Scattering from Cu, Ag, Sn, and Pb compared with that from equal masses of Al.			
Approximate wave-length.	Approximate absorbability $(\mu/\rho)_{Al}$ .	$\frac{(\sigma/\rho)_{Cu}}{(\sigma/\rho)_{Al}}$	$\frac{(\sigma/\rho)_{Ag}}{(\sigma/\rho)_{Al}}$	$\frac{(\sigma/\rho)_{Sn}}{(\sigma/\rho)_{Al}}$	$\frac{(\sigma/\rho)_{Pb}}{(\sigma/\rho)_{Al}}$
$96 \times 10^{-8}$ cm.	13	...	...	5.6	...
91 "	11	...	3.66	...	...
63 "	3.3	2.5	..	...	11.2
59 "	2.8	...	...	...	11.5
585 "	2.7	2.3	...	...	...
52 "	1.8	2.1	...	...	9.0
47 "	1.3	1.9	...	...	5.8
43 "	.95	1.5	...	...	4.4
38 "	.65	1.05?	...	...	2.85
316 "	.37	1.12	...	...	2.65
314 "	.36	1.07	1.37	...	2.1
311 "	.35	...	...	1.47	...
306 "	.33	...	1.25	...	1.9
305 "	.32	1.05	...	...	1.7

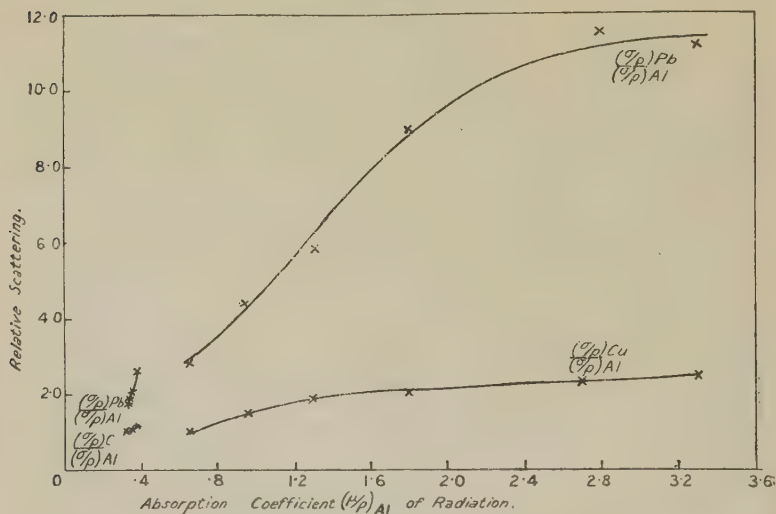
These indicate that the lighter the element the smaller is the variation of scattering with wave-length. Though the variation in aluminium itself was stated to be small, it was not given in the paper referred to.

Also within the range of wave-lengths employed, apparent discontinuities in the absorption curves for both copper and aluminium have more recently suggested the emission of J characteristic radiations (Barkla and Miss White)\*. By plotting from Table I. "relative scattering" as obtained by Barkla and Miss Dunlop against absorption coefficient, it will be seen that an irregularity appears in the same region of wave-lengths as that in which discontinuities in absorption

\* Phil. Mag. Oct. 1917.

were later found (fig. 1). No importance was attached to these when the scattering experiments were made—indeed, one of the values was queried because of its departure from the otherwise perfectly general law of increase of scattering with wave-length. The results were not examined till the whole series of observations was completed, and war work prevented a return to the experiments in search of the cause of the apparent irregularity.

Fig. 1.



[Recent examination of results of experiments upon which the conclusions shown in Table I. were based, shows that the irregularity was *largely* due to a step or discontinuity in the absorption curves, exactly as shown later in a paper by Barkla and White.]

We therefore determined to investigate more closely the scattering from copper, aluminium, and the lighter elements oxygen, carbon, and hydrogen (together) in filter paper. As in the previous investigations referred to, the homogeneous beams initially tried were not sufficiently intense for the measurements required, nor was the apparatus available for producing them. We used primary beams direct from the X-ray tube, and obtained the variation of wave-length by adjustment of gas pressure in the tube, the potential applied to the tube, and by filtering the radiation. Sheets of the above substances were placed in turn in the

primary radiation with the surface making an angle of  $45^\circ$  with the axis of the beam. The intensity of the scattered radiation proceeding in a direction at right angles to the primary radiation from the second face of the sheet was studied. In a comparison of the intensities of primary and secondary beams, errors due to absorption were thus avoided, or rather minimized, as both beams passed through the same thickness of radiator. The intensities of these two beams were measured by similar electroscopes placed in their path. The correction for scattering from air and all stray effects was found from similar observations taken in the absence of the radiating sheets.

From substances of higher atomic weight the scattered radiation or any possible characteristic radiation emerges only from a much smaller mass of radiating material owing to its greater absorbing power; hence the whole effect in the secondary electroscope is much smaller. More sensitive electroscopes in which the ionization was produced in sulphur dioxide were used to measure the scattered radiation, and two electroscopes were used in the primary beam for convenience in measuring the absorption coefficient of the radiation. The range of wave-lengths over which reliable readings could be obtained was, however, much shorter than in the experiments upon paper.

TABLE II.  
Scattering from Paper.

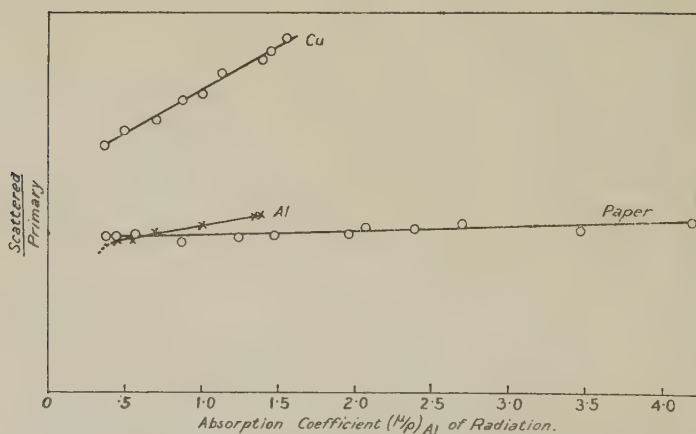
Absorption Coeff. of Radiation ( $\mu/\rho$ ) <sub>Al</sub> .	Observed Ratio of intensities, Secondary/Primary.	Stray effect.	Corrected Ratio of intensities, Secondary/Primary.
2.70	$\cdot 243 \times x$	$\cdot 022 \times x$	$\cdot 221 \times x$
2.41	$\cdot 235$	$\cdot 022$	$\cdot 213$
2.07	$\cdot 228$	$\cdot 013$	$\cdot 215$
1.48	$\cdot 228$	$\cdot 019$	$\cdot 209$
1.27	$\cdot 225$	$\cdot 019$	$\cdot 206$
0.87	$\cdot 221$	$\cdot 020$	$\cdot 201$
0.55	$\cdot 232$	$\cdot 025$	$\cdot 207$
0.44	$\cdot 233$	$\cdot 028$	$\cdot 205$
0.40	$\cdot 235$	$\cdot 032$	$\cdot 203$
4.20	$\cdot 243$	$\cdot 020$	$\cdot 223$
3.47	$\cdot 238$	$\cdot 022$	$\cdot 216$
1.97	$\cdot 235$	$\cdot 023$	$\cdot 212$

The results for paper are given in Table II., in which column 1 gives the absorption coefficient for the radiation,



column 2 the observed intensity of secondary radiation compared with intensity of the primary, and column 4 the corrected values after subtraction of the stray effect given in column 3. The corrected results are exhibited graphically in fig. 2, where the ratio of the intensities of the scattered and the primary radiations—so measured and in arbitrary units—is plotted against absorptivity of radiation ( $\mu/\rho$  in Al). Over this range  $\mu/\rho = \cdot 4$  to 4, the variation of scattering is very slight, but there is a small increase of scattering with wave-length amounting altogether to about 10 per cent. The increase cannot with certainty be distinguished from a linear one, though it is possible, indeed probable from other considerations, that the greater variation is at the two ends

Fig. 2.



of the range\*. These results of experiments upon paper are of considerable interest for the emission of even a feeble characteristic X-radiation from one of the constituent elements, or a small rapid change in the scattering would have been apparent in the form of the curve (fig. 2).

There is no evidence here of the emission of a characteristic radiation or characteristic radiations from paper. The excitation of such a radiation in appreciable intensity would be apparent as a superposed fluorescent (characteristic) radiation for wave-lengths less than a certain critical value—*i. e.*, passing from right to left of fig. 2, there would be a sudden rise in the curve for paper at a value for  $\mu/\rho$

\* This was found in the case of aluminium at the high-frequency end, just beyond the limits of the observations given in this paper.

in Al of about  $\cdot 8$ , from Barkla and White's absorption experiments. There is no suggestion of this. It should, however, be pointed out that the rise to be expected from the absorption experiments is an exceedingly small one, for the change of absorption observed was not greater than a 15 per cent. rise, and in light elements only a small fraction of the energy absorbed has been found to be re-emitted as characteristic radiation (of series K). Thus a rise of an unknown but only a small fraction of 15 per cent. might reasonably be expected. As we could not be certain of detecting a variation of less than about 3 per cent., the evidence for or against it is not conclusive. The fraction of energy transformed into fluorescent (characteristic) radiation is, however, greater in elements of higher atomic weight; a characteristic radiation should therefore be more easily detected in these. From the experiments on aluminium it was evident there was a much greater change of scattering with wave-length. That this was not due to a superposed soft characteristic radiation, either from aluminium or from a small quantity of impurity (Fe) in the aluminium, was shown by placing sheets of aluminium in the paths of both primary and secondary beams. This absorbed the FeK radiation almost completely, yet a gradient of the same order of magnitude was found. The results are given in Table III. and plotted in fig. 2.

TABLE III.

## Scattering from Aluminium.

Absorption Coefficient of Radiation ( $\mu/\rho$ ) <sub>Al</sub> .	Ratio of Intensities, Secondary/Primary.
1.39	$\cdot 470 \times y$
1.35	$\cdot 465$
1.01	$\cdot 439$
$\cdot 69$	$\cdot 416$
$\cdot 56$	$\cdot 404$
$\cdot 463$	$\cdot 399$
$\cdot 46$	$\cdot 400$

In the experiments on the scattering from copper it was of course necessary, as in previous experiments, to eliminate the copper K radiation. This was effected by placing aluminium  $\cdot 058$  cm. in thickness in the path of both primary and secondary beams after emergence from the radiator.

As shown in fig. 2 the variation of scattering from copper was found to be greater still than from aluminium. The same degree of accuracy cannot, however, be claimed for these readings, as a slight change of angle through absorption effects makes a considerable change in the slope or apparent variation of scattering with wave-length. Further experiments would need to be made to get the exact rate of variation: the results already obtained serve our immediate purpose.

As in the case of the paper there was no evidence either in the aluminium or copper curve of a discontinuity such as might have been expected to appear if characteristic radiation had been emitted. Thus these experiments give not the slightest evidence of the emission of a characteristic radiation of higher frequency than that of the K series.

As there was no appreciable characteristic X-radiation in any of these secondary beams, the radiations studied were presumably scattered X-radiations. Further evidence for this is, however, given below.

Any small transformation, unless constant in amount, might equally have been expected to show itself by a gradual change in the slope of the curves, for a change in absorptability is accompanied by a change in ionizing power such as would have affected the measurements of the secondary radiation. This is not indicated in the curve for paper.

These results thus confirm the previous conclusions regarding relative rates of variation of scattering with wave-length obtained by direct comparison of various substances with aluminium. They also show with accuracy the small variation of scattering coefficient in the case of carbon, oxygen, and hydrogen, a greater variation in aluminium, and greater still in copper; in addition, they show the variation to be a gradual one over the range of wave-lengths employed.

*Relative Intensities of Scattered Radiation from  
various Substances.*

So far these experiments gave only rates of variation of scattering with wave-length of the radiation. Other experiments were undertaken to determine the relative intensities of the radiation scattered from equal masses of various substances.

In the experiments of Barkla and Miss Dunlop comparisons had been made between the scattering from various substances

with the scattering from aluminium. A more exact knowledge of the rate of variation of scattering with wave-length is obtained by comparison with the scattering from paper in which the variation is very small.

In these experiments two substances were compared directly. The thicknesses were adjusted so that the two radiating sheets absorbed the same proportion of the primary beam. As there were no marked selective effects over the wave-lengths employed, the correction for absorption was avoided. Thus a comparison was made of the scattering from the two radiators alternately placed in the same position in the primary beam. Dividing by the relative masses of the two we have the relative scattering by equal masses of the two substances. An absorbing sheet of aluminium was placed in the path of both the primary and secondary beams, and the ratio was found to be as nearly as measurable identical—showing the difference in intensity was not due to the superposition of soft characteristic radiation from any one of them. It showed too that if a change of absorbability was produced by re-emission (scattering), it was at least approximately the same for the radiation from paper, aluminium, and copper. For radiation of absorbability in aluminium ( $\mu/\rho$ ) equal to about 1.3, the relative intensities from aluminium and paper were found varying from 1.07 to 1.13, giving an average value of 1.1 (see Table IV.). In

TABLE IV.  
Relative Scattering Al/Paper.

Absorption Coefficient $\mu/\rho$ in Al.	Thickness of Al before Electroscopes.	Relative Scattering Al/Paper.
1.14	0 cm.	1.07
1.24	0	1.08
	.036	1.09
1.32	0	1.07
	.036	1.12
1.39	0	1.13
	.036	1.14
1.45	0	1.08

the experiments on the relative scattering from copper and aluminium, it was necessary, as in the previous experiments, to absorb the copper "K" radiation by placing aluminium .058 cm. in thickness before each electroscopes; the radiation

scattered from aluminium was, of course, observed under identical conditions (see Table V.). The ratio of intensities

TABLE V.  
Relative Scattering Cu/Al.

Absorption Coefficient $\mu/\rho$ in Al.	Thickness of Al before Electroscopes.	Relative Scattering Cu/Al.
·795	·204 ? cm.	1·73 ?
·858	·180	1·70
1·00	·180	1·88
1·22	·058	1·74
1·26	·056	1·88
1·26	·204	1·88
1·33	·058	1·86
1·5	·056	1·88

from copper and aluminium was very consistently about 1·87 when the absorbability of the radiation was approximately as before ( $\frac{\mu}{\rho_A} = 1·3$ ). This, again, agrees very well with the relative value 1·9 found by Barkla and Miss Dunlop by another method (see Table I.).

The results of these comparisons were then used to give the relative heights of the curves in fig. 2 by adjustment of the relative values for radiation of absorption coefficient = 1·3.

*Comparison of Absorption Coefficients of Primary  
and Secondary Rays.*

As the experiments described furnished no evidence of the emission of a characteristic radiation of higher frequency than that of the K series, we proceeded to make a direct comparison of the penetrating powers of the primary and secondary X-radiations from light elements. In the earliest experiments by one of us\*, when the conditions were made as simple as possible, and only moderately low frequencies were used, no difference in penetrating power of primary and scattered radiations could be found directly—yet some evidence of transformation was observed.

Later, differences were observed which appeared to indicate either the admixture of an unidentified fluorescent (characteristic) radiation, or just possibly a transformation in scattering.

\* Phil. Mag. May 1903, June 1904.



These effects, however, were very variable in magnitude, and the further investigation of them was left for experiments of the kind described in the first part of this paper.

While the effect of variation of scattering with wave-length, and the emission of an X-radiation excited in the absorbing substance by the electrons constituting the secondary corpuscular radiation, must not be lost sight of, the principal source of uncertainty lies not in these but in the difficulty of obtaining a true measure of an intense primary beam of considerable cross-section. While it would be premature to give any decided opinion upon the nature of apparent transformations before the phenomena have been more fully investigated, our preliminary results sufficiently limit the possibilities to justify publication at this stage.

When thin sheets of paper were used as the scattering substance—in which case trouble due to differential absorption of primary and secondary beams in the radiating substance was avoided—experiments showed that the scattered was identical with the primary radiation throughout almost the whole range of wave-lengths experimented upon. This is seen from a comparison of the absorption coefficients of primary and secondary radiations, as given in Table VI., columns 1 and 2 respectively, and plotted in fig. 3. The simple law of equality where there is infinite possibility of variation is sufficient. Thus there is no doubt under certain conditions of an accurate equality of penetrating power of the scattered and primary radiations through a long range of wave-lengths included in these experiments, though this does not include the wave-length of the J absorption discontinuity of Barkla and White.

This conclusion is confirmed through a longer range of wave-lengths, in this case including the J absorption discontinuity, and without even the evidence of possible exception by the experiments on the variation of scattering with wave-length, the results of which are shown in fig. 2.

Thus under some conditions it is possible to obtain a secondary X-radiation from paper (O, C, and H) which is purely scattered radiation, without change in penetrating power and without admixture of fluorescent X-radiation, throughout the whole long range of wave-lengths. [The limits of experimental error are about 3 per cent. in intensity, or 3 per cent. in absorption coefficient.]

Under other conditions decided differences have been observed between the absorption coefficients of primary and secondary radiations, but only in the region of higher frequencies. This is shown by a comparison of the values

TABLE VI.

Absorption Coefficients (approximate) of Primary and Secondary Radiation (Scattered) from Paper.

Primary Radiation ( $\mu/\rho$ in Al).	Secondary Radiation ( $\mu/\rho$ in Al).	
Column (1).	Column (2).	Column (2x).
10.3	10.4	
8.46	8.76 (approx.)	
6.74	6.83	
6.39	6.51	
6.34	6.35	
6.23	6.32	
6.07	6.04	
6.03	6.16	
5.86	5.92	
5.73	5.72	
5.31	5.36	
5.18	5.12	
4.91	4.94	
4.88	4.85	
4.42	4.43	
3.27	3.20	
3.14	...	3.45
2.46	...	2.90
2.33	...	2.83
2.07	2.18	
2.05	2.14	
1.90	2.00	
1.83	...	2.02
1.51	...	1.94
1.47	...	1.83
1.16	1.21	
1.15	1.20	
0.65	...	0.90?
0.58	...	0.82?
0.54	...	0.70?

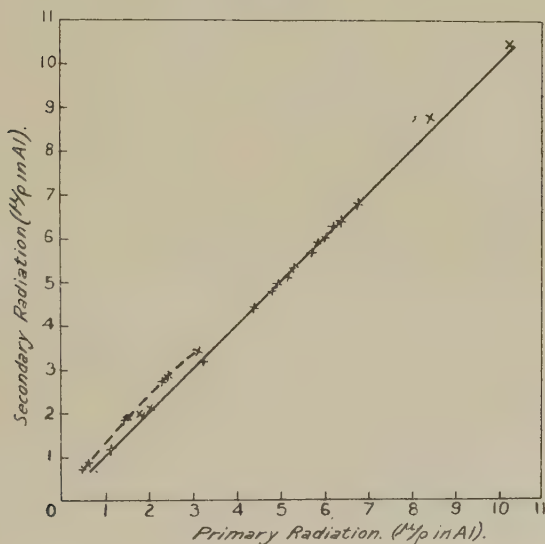
in columns 1 and 2x, obtained at other times with the same experimental arrangement. The corresponding points are connected by the broken line of fig. 3. It is important to notice that if such transformations had occurred during the experiments on the variation of scattering with wave-length we might have expected quite a large deviation from the simple relationship shown in fig. 2, in the curve for paper, for a change in absorbability normally implies a change in ionizing

power which would have been detected in the ionization produced in the secondary electroscope.

This fact compels us to consider very critically the transformations which have been observed in many experiments and have recently been recorded by Crowther and by A. H. Compton.

It should also be observed that no step in the absorption curve for paper has ever been found of anything like the magnitude required to explain Crowther's results.

Fig. 3.



Thus, if appreciable transformation does occur, it is only under special conditions which have not yet been identified. Moreover, we quite realise that experiments indicating such transformation have not been made under conditions which entirely eliminate the possibility of error or of an explanation in terms of certain phenomena observed in our experiments upon the J absorption discontinuities. The transformed radiations, if true transformation in the paper be established, are not subject to the same laws as those governing the fluorescent radiations of series K, L, and M.

We shall, however, have more to say concerning the J radiations.

*Summary.*

Further experiments on the scattering of X-rays show that even the light elements hydrogen, carbon, and oxygen (together) show a very slight increase of scattering with wave-length. Over a considerable range, however, the outstanding feature is the constancy of the scattering coefficient—as compared with absorption and other coefficients.

In aluminium, and more markedly in copper, the scattering increases more rapidly with wave-length.

The relative intensities of scattering by hydrogen, carbon, and oxygen (in paper), aluminium, and copper have been obtained by another method, and the results for the last two—the others have not been previously determined—agree very closely with those given by Barkla and Miss Dunlop.

Over a considerable range of wave-lengths no difference between the absorption coefficient of the primary radiation and of the secondary X-radiation (scattered) from paper has been detected. The marked differences sometimes appearing when the higher frequency radiations are used have not been observed under conditions justifying any definite conclusion regarding their origin. Further experiments are, however, being conducted, as the investigation does not appear to have completely exhausted all possibilities of an explanation in terms of the simple laws of scattering, fluorescence, and absorption, which have already found general acceptance.

These investigations have not furnished the evidence of the existence of J characteristic radiation which so many results led us to expect. Further communications on this subject will, however, shortly be made.

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LXXIII. *Some Properties of Resonance Radiation and Excited Atoms.* By K. T. COMPTON, Ph.D., Professor of Physics, Princeton University\*.

IN accounting for spectral lines, for their variation in intensity with different conditions of excitation and for phenomena of cumulative ionization, attention has recently been drawn to the existence of atoms and molecules in various conditions of partial or complete ionization. In particular, that type of excited atom which results from the first displacement of an electron from its most stable position

\* Communicated by the Author.

is of great interest because it may be produced in such quantity, by proper excitation, as to give to the gas properties which are quite different from those of the normal unexcited gas. Such properties include the appearance of new lines in the absorption spectrum, new types of resonance radiation, band spectra associated with molecular combinations between excited atoms and increased ease of ionization. A study of these properties has led to discoveries of importance in the theories of atomic structure and of radiation.

In recent papers \* the writer discussed the two ways in which atoms may be put into the excited, or partially ionized, state. It was shown that, in a discharge-tube, the number of atoms in the excited state as a result of a preceding electronic impact is exceedingly small in comparison with the number in this state through absorption of the resonance radiation excited by electronic impacts against neighbouring atoms, unless the gas pressure is very low. This fact suggests an interpretation of experiments of Franck and Knipping † and of Kannenstine ‡ on metastable helium.

The conception of partial ionization by resonance radiation is as follows :—When an electron whose velocity exceeds that gained by falling through the resonance potential  $V_r$  collides with an atom, it displaces an electron from the normal to the next outer orbit, making an excited atom. After a time-interval, of average duration  $\tau$ , this electron falls back to its original position and emits a quantum  $h\nu$  of resonance radiation. Any other normal atom is capable of absorbing this radiation, and re-emitting it after an average interval  $\tau$ . The radiation is thus passed on from atom to atom, putting one after another into the excited state, until it finally escapes from the gas or is degraded into light of lower frequency as a result of a temporary chemical combination made by an atom while in the excited state.

Formally, if not physically, the passage of resonance radiation through a gas may be treated as a problem in diffusion; for the intensity of a beam of light is reduced by absorption or resonance scattering according to the equation  $I = I_0 e^{-\alpha x}$ , and the number of molecules passing through a gas in a parallel beam falls off according to the similar equation  $N = N_0 e^{-x/l}$ . Thus the reciprocal  $l/\alpha$  of the absorption coefficient is analogous to the mean free

\* Phys. Rev. xx. (1922); Phil. Mag. xliii. p. 531 (1922).

† Zeit. f. Phys. i. p. 320 (1920).

‡ Astrophys. Journ. lv. p. 345 (1922).

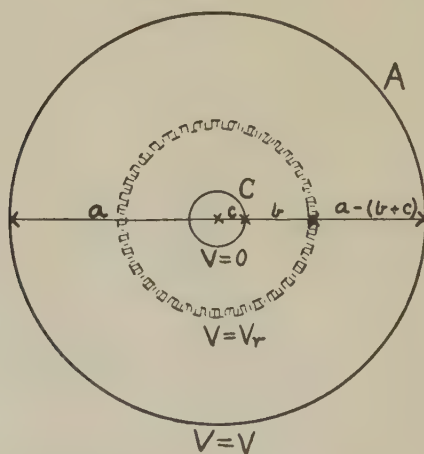


path  $l$ . It follows that the average speed with which a quantum of radiant energy moves through a gas equals  $\frac{l}{\alpha\tau}$ , or its mean free path  $l/\alpha$  divided by the average time  $\tau$  during which it is absorbed by the atom at the end of its path. Thus we may apply the diffusion equation from the Kinetic Theory of Gases, and put

$$\iint \frac{1}{3} \frac{1}{\alpha} \frac{1}{\alpha\tau} \frac{dN'}{dn} dS = - \iiint R dx dy dz, \quad . \quad . \quad (1)$$

in which  $N'$  is the number of excited atoms per unit volume at any point,  $n$  is the outward normal to the element of surface  $dS$ , and  $R$  is the rate of production of new quanta

Fig. 1.



per unit volume. The surface integral gives the rate of "diffusion" of quanta across the closed surface, and the volume integral gives the rate of production within it. By applying this equation we can calculate the concentration  $N'$  of excited atoms at any point in the gas, as is done for coaxial cylindrical electrodes in the following treatment.

Electrons from the hot cathode  $C$  collide inelastically with atoms, giving rise to resonance radiation, in the shaded layer. If the length  $f$  of the electrodes is large compared with the distance  $(a-c)$  between them, this resonance radiation diffuses only inward and outward, ultimately being absorbed by the electrodes.

If  $n$  electrons per second leave the cathode, and if  $P$  is

the fraction of atoms in the excited state in the shaded layer, then  $n(1-P)$  is the number of quanta of resonance radiation produced in this layer per second.  $n_a(1-P)$  quanta diffuse outward to the anode and  $n_c(1-P)$  diffuse in toward the cathode. If we apply equation (1) to a cylindrical surface of radius  $r$ , lying outside the shaded layer, we have

$$\frac{1}{3\alpha^2\tau} \frac{dN'}{dr} (2\pi r f) = -n_a(1-P);$$

whence

$$N'_a = -\frac{3\alpha^2\tau n_a(1-P)}{2\pi f} \log r + C_a$$

is the value of  $N'$  at any point outside the shaded layer. If the reflecting power of the anode for resonance radiation is negligibly small,  $N'_a = 0$  when  $r = a$ , whence

$$N'_a = \frac{3\alpha^2\tau n_a(1-P)}{2\pi f} \log \frac{a}{r}.$$

Similarly, at points inside the neutral surface,

$$N'_c = \frac{3\alpha^2\tau n_c(1-P)}{2\pi f} \log \frac{r}{c}.$$

But  $n_a + n_c = n$  and at the layer  $r = (b+c)$  we have  $N'_a = N'_c = N'$ . It is thus easily shown that

$$N'_a = \frac{3\alpha^2\tau n(1-P)}{2\pi f} \frac{\log \frac{b+c}{c}}{\log \frac{a}{c}} \log \frac{a}{r},$$

$$N'_c = \frac{3\alpha^2\tau n(1-P)}{2\pi f} \frac{\log \frac{a}{b+c}}{\log \frac{a}{c}} \log \frac{r}{c}.$$

Integrating  $N'_a$  over the volume of gas outside the shaded layer and  $N'_c$  over the volume within it, and adding, we obtain

$$N = \frac{3\alpha^2\tau n(1-P)}{4 \log \frac{a}{c}} \left\{ [a^2 - (b+c)^2] \log \frac{b+c}{c} - [(b+c)^2 - c^2] \log \frac{a}{b+c} \right\} \quad (2)$$

as the total number of excited atoms between the electrodes in the gas.

In deriving this expression, we assumed that the reflecting power of the electrodes for resonance radiation is negligible. If we had included it in our calculations, the constant  $C_a$  would have taken a more complicated value and the value of  $N$  in equation (2) would have been increased. The error thus introduced is relatively small, however, especially for resonance radiation of short wave-lengths.

In equation (2)  $a$  and  $c$  are dimensions of the apparatus, and  $b$  depends on the distribution of potential between the electrodes. The fraction  $P$  of atoms in the excited state at the shaded layer is usually small compared with unity\*, although it is possible that it may approach unity in very intense arcs.  $\tau$  is the average interval between absorption and emission of a resonance quantum of energy by an atom. According to classical dynamics this is the reciprocal of the electromagnetic damping constant, or

$$\tau = \frac{3mc^3}{8\pi^2 e^2 \nu^2}, \quad \dots \dots \dots (3)$$

where  $e$  and  $m$  are the electronic charge and mass,  $c$  is the velocity of light, and  $\nu$  is the frequency. From this we calculate  $\tau = 2(10)^{-8}$  sec. for the  $H_a$  line. For the resonance radiation in helium we calculate  $\tau = 1.7(10)^{-10}$  sec. The most accurate experimental determinations of  $\tau$  are probably those of Wien†, who found  $\tau = 2.35(10)^{-8}$  sec. for  $H_\beta$  and  $H_\gamma$ , and about the same value for lines and bands of oxygen and nitrogen. These are of the order of magnitude to be expected from equation (3), but do not accurately confirm it, since very little variation of  $\tau$  with frequency  $\nu$  was found. The direct experimental data are meagre, and give no information as to whether  $\tau$  is the same for resonance as for other radiation, or whether the approximate agreement with equation (3) in the region of the visible spectrum is a mere coincidence.

The scattering coefficient  $\alpha$  is very difficult to measure, since it is only the central monochromatic portion of a resonance line which is scattered by the gas. Lamb‡ gives a very general law of resonance scattering, applicable if every molecule possesses an oscillator of resonance frequency. Under this condition his law may be expressed in the form

$$\frac{p}{\alpha} = \frac{1.54}{N\lambda^2}, \quad \dots \dots \dots (4)$$

\* K. T. Compton, *loc. cit.*

† *Ann. d. Phys.* lx. p. 597 (1919), p. 229 (1921); also Dempster, *Phys. Rev.* xv. p. 138 (1920).

‡ *Camb. Phil. Soc. Trans.* xviii. Stokes Commemoration (1900).

where  $p$  is the pressure in millimetres,  $N$  is the number of scattering atoms per  $\text{cm}^3$  at 1 mm. pressure, and  $\lambda$  is the wave-length of the resonance radiation. Equation (4) gives a theoretical upper limit for  $\alpha$ . This limit might be approached if there were entire absence of Doppler shifts, and if every atom were at every instant possessed of an oscillator capable of responding to resonance frequency.

Actually  $\alpha$  has not been measured for any resonance line except mercury 2536, owing to experimental difficulties. For this line Wood's results\* yield a value  $p/\alpha = 6.75(10)^{-4}$  cm. as the "mean free path" of the resonance radiation at 1 mm. pressure. Equation (4) gives a value  $6.6(10)^{-8}$  cm., about 10,000 times smaller than the observed value, assuming every atom to be effective in scattering. This discrepancy must be due (1) to inability to get strictly monochromatic light, (2) to the Doppler effect of the moving atoms, and (3) to the fact that an atom probably has the internal configuration to permit its absorption of a given frequency only at repeated intervals. For helium, equation (4) gives  $p/\alpha = 1.16(10)^{-6}$  cm.

It is evident that our present knowledge of  $\alpha$  and  $\tau$  is insufficient to permit the use of equation (2) for anything but the very roughest estimates of the number  $N$  of excited atoms to be expected in an apparatus on the present theory. Nevertheless, some interesting suggestions can be made.

Kannenstine† found, by use of an alternating voltage of variable frequency to produce an arc by thermionic bombardment in helium, that excited helium atoms could be detected as long as 0.0024 sec. after the exciting voltage had been removed. If excited atoms are produced only, or mainly, as a direct result of electron impacts, then 0.0024 sec. must be time during which *individual atoms* persist in the excited state. It must be, approximately, the value of  $\tau$ , and therefore about 100,000 times larger than any value of  $\tau$  measured directly and 15,000,000 times the classical value predicted by equation (3).

If, on the other hand, excited atoms may also be formed by absorption of resonance radiation, the interpretation of Kannenstine's work is quite different and more in harmony with our other evidence. 0.0024 sec. represents the time required for the resonance radiation to escape from the gas, by the mechanism discussed in this paper, and this may include the time of many thousands of absorptions and re-emissions. An accurate solution of the relation between

\* 'Recent Researches in Physical Optics.'

† *Loc. cit.*

0.0024 sec. and  $\tau$  may be obtained by a Fourier series method, analogous to that used for the escape of heat from a cylinder whose initial distribution of temperature is given. The following simpler method is sufficiently accurate for the present purpose.

At the instant the voltage is shut off,  $n(1-P)$  is the rate of production of new quanta, and hence the rate of escape of quanta from the gas, and equation (2) gives the total number of quanta  $N$  in it. Hence  $N/10n(1-P)$  is the number of seconds  $t'$  required for 1/10 of the quanta to escape from the gas, if the loss were to continue at the initial rate during this interval, which is not quite true. If we neglect this relatively small error, and substitute for  $N$  from equation (2), we have

$$t' = \frac{3\alpha^2\tau}{40 \log \frac{a}{c}} \left\{ [a^2 - (b+c)^2] \log \frac{b+c}{c} - [(b+c)^2 - c^2] \log \frac{a}{b+c} \right\}. \quad (5)$$

The true value of  $t'$  must be a little greater than this.

It is difficult to estimate the ratio by which the number of excited atoms decreased during the time 0.0024 sec. in Kannenstine's experiments. He would certainly have detected as many as 1/10 the initial number, and would probably not have detected them if reduced to 1/1000. These reductions correspond approximately to time intervals of  $22t'$  and  $66t'$ , respectively. Probably about 1/100 of the initial number would have been detected, corresponding to about  $44t'$ . This is, then, the quantity to which his time of 0.0024 sec. refers. We may, therefore, to at least the right order of magnitude, put

$$0.0024 = \frac{33\alpha^2\tau}{10 \log \frac{a}{c}} \left\{ [a^2 - (b+c)^2] \log \frac{b+c}{c} - [(b+c)^2 - c^2] \log \frac{a}{b+c} \right\}. \quad (6)$$

From this equation,  $\tau$  could be found if  $\alpha$  were known, and *vice versa*.

In an ordinary arc, stimulated by electronic bombardment, we may take  $a=1$  cm.,  $c=0.02$  cm. as approximate dimensions of the apparatus. In the arc, with a positive space-charge around the cathode,  $b$  appears to be of the order of the electronic mean free path in the gas\*, and hence about 0.1 cm. in Kannenstine's experiments. This

\* Compton, Phys. Rev. (in print).



makes the term in braces in equation (6) equal to about 1.8, and any reasonable values of  $a$ ,  $b$ , and  $c$  lead to this order of magnitude. We have, then,

$$\alpha^2\tau = 0.0016 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

as a rough expression of the relation between  $\alpha$  and  $\tau$  given by Kannenstine's experiments.

If we assign to  $\alpha$  the upper limiting value, as given by equation (4), we find  $\tau = 2.2(10)^{-15}$  sec. If we suppose that  $c$  is actually about 10,000 times smaller than the limiting value for helium, as it appears to be for mercury, we have  $\tau = 2.2(10)^{-7}$  sec. If we assume that the resonance radiation is scattered at the same rate in helium as in mercury, we have  $\tau = 0.75(10)^{-9}$  sec. Owing to its simple structure, we should expect a larger proportion of helium than of mercury atoms to be in a condition capable of absorbing resonance radiation at any instant, so that we should expect the estimate  $\tau = 2.2(10)^{-7}$  sec. to be too large. *The most reasonable interpretation of Kannenstine's experiments would seem, therefore, to indicate a value of  $\tau$  for helium which is not noticeably larger than for other substances.*

Unsatisfactory as this treatment is, owing to uncertainty regarding the value of  $\alpha$ , it at least serves to emphasize the tendency of resonance radiation to remain imprisoned within a gas for a time which may be enormous in comparison with the time  $\tau$  of imprisonment within individual atoms.  $c$  is proportional to the gas pressure, so that this imprisoning effect should become more important at high pressures, and become negligible below pressures at which the mean free path of the radiation  $1/\alpha$  exceeds the dimensions of the apparatus. We should expect Kannenstine's time-interval 0.0024 sec. to increase with increasing gas pressure, although this would be partially masked by the accompanying decrease in  $b$  and, possibly, by change in the space-charge conditions. He states that the persistence of metastable atoms was more marked at higher than at lower pressures, although no noticeable difference in time of persistence was found.

The general idea embodied in the above treatment appears to be a necessary consequence of the nature of resonance radiation. May the resonance radiation disappear from the gas in any other manner than by diffusion out of its boundaries as discussed above? It seems certain that the resonance radiation may be "degraded" into radiation of other wave-lengths during the existence of temporary compounds with other atoms of the same, or of different gases.

Thus increase of pressure, or addition of relatively chemically active impurities, would tend to diminish the amount of resonance radiation imprisoned in the gas and, therefore, the number of excited atoms. Several facts may be accounted for by such "degradation."

Lyman \* reports spectroscopic observation of helium lines 584.4, 537.1, 522.3, and 515.7 Å, which seem to be the lines of the principal singlet series  $0S-mP$ . If this be true, the line 584.4, which corresponds to the 21.2 volt resonance potential, does not represent a transition in violation of the selective principle, and there is therefore no theoretical reason for believing the corresponding type of excited atom to have an abnormally long life  $\tau$ . Lyman also reports a queer diffuse line at 600.5 Å, corresponding to the 20.4 volt resonance potential, which is usually a stronger effect than that at 21.2 volts, and possesses stronger resonance properties. The spectroscopic character of this line would be accounted for if atoms in the 20.4 volt excited state are chemically more active than in the 21.2 volt state, resulting in the formation of temporary helium compounds. The internal forces, or the alterations in the internal energy, would alter the frequency of the radiation emitted when the atoms return to the normal state, so that this would not be resonance radiation. Since the helium compounds are very unstable, such an effect would result in a distributing of the original resonance radiation into a not very wide diffuse line or band as observed. There is independent evidence that the 20.4 volt excited atoms are chemically active †. In addition to such a diffuse line, a regular band spectrum, to which the energy of the absorbed resonance radiation might conceivably contribute, would be expected from the helium molecules—as is the case ‡.

In the presence of other gases, any of which would be chemically more active and would be photoelectrically ionized by the resonance radiation of helium, this radiation would be converted into other and longer wave-lengths more rapidly than in pure helium, both through excitation of the line spectra of impurities and band spectra of various temporary molecular combinations. This suggests an explanation of results of Franck and Knipping, who found the photoelectric effect of 20.4 volt resonance radiation to become relatively weaker as the helium was purified, and interpreted this as indicating that the return of such metastable atoms

\* Science, lvi, p. 167 (1922).

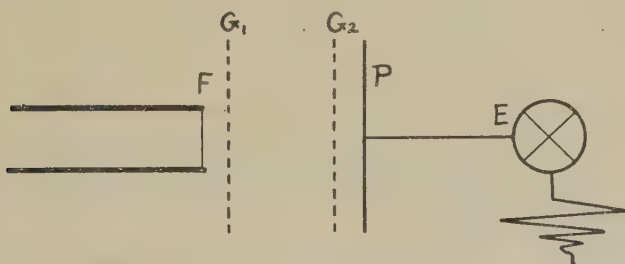
† Francke & Reiche, *Zeit. f. Phys.* 1, p. 154 (1920).

‡ Curtiss, Roy. Soc. Proc. A, ci, p. 38 (1922).

to the normal state occurred only in the presence of foreign gas molecules.

Their apparatus was of the familiar type indicated in fig. 2, the radiation resulting from electron impacts with molecules between two gauzes  $G_1$  and  $G_2$ , and being detected by the photoelectric effect on the plate P. If the radiation is strictly of the resonance type, it would tend to be absorbed in the gauzes, owing to the extremely short "mean free path" of the radiation. It is as if the wire of the gauze were a perfect absorber of molecules introduced between the gauzes and allowed to diffuse outward with a very short free path through the helium. Relatively few would reach the plate P. In the presence of impurities, on the other hand,

Fig. 2.



the resonance radiation would be transformed into non-resonance wave-lengths, and would pass freely through the openings in the gauze and produce a photoelectric effect on the plate P.

It may be suggested, furthermore, that "degradation of resonance radiation," in some such manner as is here suggested, seems necessary to account for the tendency of radiation to become adapted, as regards spectral distribution of energy, to the temperature of the medium through which it is passing. Without such degradation, the observed radiation from the sun would be more characteristic of its hot interior than is actually the case.

### Conclusion.

The present discussion, though rather speculative and dealing with quantities which are very uncertainly known, yet serves to point out certain important properties of resonance radiation, and calls attention to the bearing which these may have on the very important but elusive problem of the existence of atoms in various excited states.

It is shown (1) that the passage of resonance radiation through a gas may be treated as a problem in diffusion, (2) that the number and persistence of excited atoms depends jointly on the average time of activation  $\tau$  of an atom and the square of the scattering coefficient  $\alpha^2$  of the resonance radiation in the gas and may be calculated, and (3) that the presence of impurities may affect experiments on the properties of resonance radiation by causing its transformation into other wave-lengths to which the gas is more transparent.

*Note added with proof.*—Attention is called to a paper on this subject, by F. A. Horton and A(nn) C. Davies (Phil. Mag. xlv. p. 1140, 1922), in which conclusions similar to those in this paper are reached.

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LXXIV. *Testing the Expressions for the Longitudinal and Transverse Masses of the Electron.* By L. T. JONES and W. C. POMEROY\*.

SINCE the first work of Kaufmann† demonstrating the mass of the electron to be a function of the velocity, a very considerable amount of experimental labour has been performed by various investigators‡ to verify the law of variation. In each case the method has involved only the transverse mass, and the curves representing the various theories lie so close together that it is difficult to decide which is most nearly correct.

The first suggestion of an experimental method involving the longitudinal mass was made by Einstein§. The difficulties of the method are obviously great and no one has carried it out.

Page|| has derived an equation relating to Hull's magnetron¶ and involving the relativity expressions for both

\* Communicated by the Authors.

† *Gesell. Wiss. Gott. Nachr.* iii. pp. 90–103.

‡ Neumann, *Ann. d. Phys.* xlv. p. 529; Guye and Ratnosky, *Arch. des Sci.* xxxi. p. 293; Guye and Lavanchy, *Arch. des Sci.* xlii. pp. 286, 353, 441; Jones, *Phys. Rev.* viii. p. 52; Bucherer, *Ann. d. Phys.* xxx. p. 974; and others.

§ Einstein, *Ann. d. Phys.* xxxi. p. 583.

|| Page, *Phys. Rev.* xviii. p. 58.

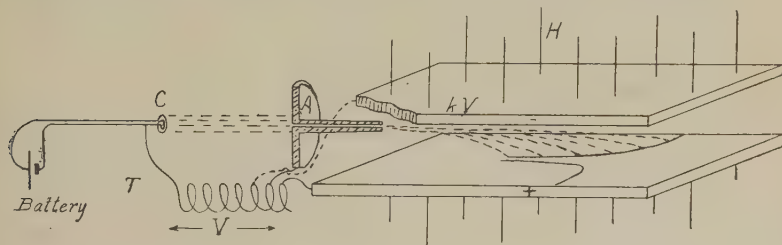
¶ Hull, *Phys. Rev.* xviii. p. 31.

masses. This equation lends itself to experimental verification, though the labour necessary is great.

The equations derived in the following pages are based on an experimental method by which the expressions for the longitudinal and transverse masses of the electron may be verified, not only for the Lorentz-Einstein electron, but for those of Abraham, Walker, and Bucherer-Langevin as well. The calculated curves of fig. 2, representing the several theories, are seen to lie twice as far apart as those encountered in previous investigations, where only the transverse mass was involved.

### *The Method.*

The essential portions of the apparatus (fig. 1) are enclosed in a vacuum maintained as high as possible. A battery heats the filament of the cathode, C. The anode, A, is



*Fig. 1.*

provided with a tube of small bore for the purpose of restricting the beam. This tube terminates midway between the two brass plates, which are horizontal. By means of the transformer, T, a high alternating potential is maintained between the cathode and anode, while a small fraction,  $k$ , of this discharge-voltage is maintained between the horizontal brass plates. The electrostatic field between these plates is thus variable and in phase with the discharge-voltage. Distortion of the field is eliminated by having the anode potential midway between the potentials of the two plates. In the discharge-chamber each electron undergoes an acceleration along the axis of the tube. When it passes through the anode and enters the deflexion-chamber it is accelerated downward by the electric field between the two plates. The total time of travel of any one electron is small compared with the time of alternation of the discharge-voltage.



A constant magnetic field,  $H$ , collinear with the electrostatic field deflects the electron beam to one side or the other of the centre line. The dotted lines indicate the electron paths when a large number of velocities are present. The more rapidly moving electrons have the shorter range. The electron beam leaves its trace on a photographic film deposited on the lower of the two horizontal brass plates. When no magnetic field is present this trace is a straight line down the centre of the plate. The two directions of the magnetic field will give two curves, one on either side of the centre line. The form of these curves will be compared with those calculated by the various theories.

*The Calculation of the Curves.*

Let

- $v$  = velocity of the electron,
- $\beta$  = the usual velocity ratio,
- $m_t$  = transverse mass of the electron,
- $m_l$  = longitudinal mass,
- $V$  = instantaneous value of discharge-voltage, e.m.u.,
- $e$  = charge of the electron,
- $c$  = velocity of light.

The work done by the discharge-voltage in carrying the electron from cathode to anode is

$$Ve = \int m_l v dv,$$

whence

$$Ve = \int_0^\beta m_l c^2 \beta d\beta. \quad \dots \quad (1)$$

*The Action of the Electrostatic Deflecting Field.*

Since the electron experiences no appreciable change in velocity while traversing the deflexion chamber, the transverse and longitudinal masses may be considered constant during that interval, having the values corresponding to its particular velocity. The horizontal velocity and the vertical acceleration are thus constant in this region.

The downward force of the electrostatic field on the electron is

$$\frac{kVe}{2d} = m_t a, \quad \dots \quad (2)$$

- where  $kV$  = instantaneous voltage between plates,
- $2d$  = distance between the two plates,
- $a$  = the acceleration of the electron.

Since the electron enters midway between the two plates, the distance of fall is

$$d = \frac{1}{2}at^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the range is

$$R = vt = c\beta t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where  $t$  is the time of travel from anode to film.

From equations (2), (3), and (4) we have

$$Ve = \frac{4m_e d^2 c^2 \beta^2}{kR^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

### *The Action of the Magnetic Field.*

During the passage of the electron through the electrostatic field it is also subject to the action of a uniform and constant magnetic field,  $H$ . Its path is bent, in the horizontal plane, into a circle of radius  $r$ . The force of the magnetic field on the electron is

$$Hev = \frac{m_e v^2}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If the deflexion,  $y$ , due to the magnetic field is small, it is represented with sufficient accuracy by

$$y = \frac{R^2}{2r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From equations (6) and (7), placing  $v = c\beta$ , we obtain

$$He = \frac{2m_e y c \beta}{R^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In equations (1), (5), and (8) the various expressions for  $m_i$  and  $m_e$  are to be introduced, yielding expressions for the coordinates  $R$  and  $y$ .

### *The Lorentz-Einstein Theory.*

The Lorentz-Einstein \* expressions for the two masses are

$$m = m_0(1 - \beta^2)^{-1/2},$$

$$m = m_0(1 - \beta^2)^{-3/2}.$$

\* H. A. Lorentz, 'Theory of Electrons.'



Assuming values of  $\beta$  we may calculate the corresponding values of the coordinates  $R$  and  $y$  by the use of (10) and (11). These values are given in columns 2 and 6 of Table I. The curve plotted from these values is indicated in fig. 2.

TABLE I.

R cm.					y cm.			
	Lorentz-Einstein.	Abraham.	Bucherer-Langevin.	Walker.	Lorentz-Einstein.	Abraham.	Bucherer-Langevin.	Walker.
1	28.24	28.16	28.27	28.26	8.17	8.17	8.22	8.20
2	28.14	28.07	28.18	28.21	3.994	3.99	4.04	4.02
3	27.96	27.90	28.06	28.11	2.559	2.60	2.63	2.59
4	27.69	27.81	27.88	27.96	1.809	1.87	1.85	1.84
5	27.31	27.51	27.62	27.75	1.331	1.40	1.43	1.37
6	26.83	27.15	27.27	27.47	.988	1.066	1.10	1.03
7	26.19	26.62	26.78	27.08	.720	.806	.844	.766
8	25.30	25.91	26.07	26.51	.494	.590	.623	.539
9	23.96	24.83	24.93	25.61	.286	.390	.410	.324

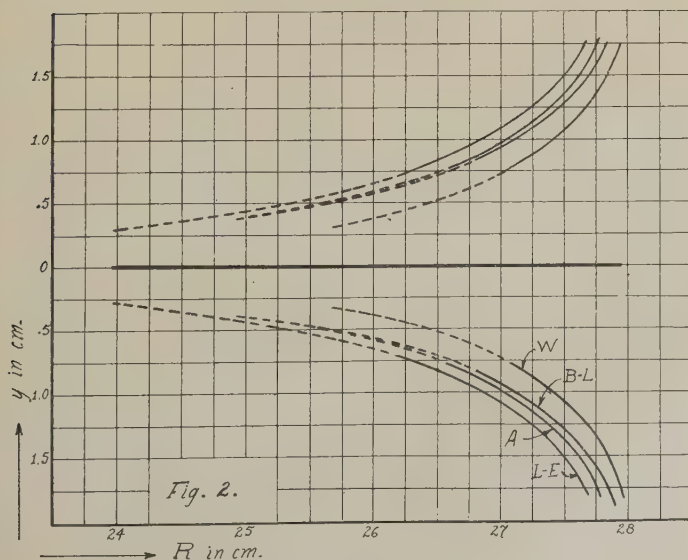
$$d=1 \text{ cm.}$$

$$k=.01.$$

$$H=3.5 \text{ gauss.}$$

$$e/m_0=1.765 \times 10^7.$$

$$c=3 \times 10^{10} \text{ cm./sec.}$$



*Walker's Theory.*

In meeting certain objections that have been raised to Lorentz's derivation of the mass expressions, Walker\* has derived values which, as corrected by Schott†, are :

$$m_t = m_0(1 + \frac{1}{6}\beta^2)(1 - \beta^2)^{-1/2},$$

$$m_l = m_0(1 - \frac{1}{5}\beta^2)(1 - \beta^2)^{-3/2}.$$

Substituting this value of  $m_l$  in equation (1) we have

$$Ve = \int_0^\beta m_0 c^2 (1 - \frac{1}{5}\beta^2)(1 - \beta^2)^{-3/2} \beta d\beta. \quad (14)$$

Integrating by parts this becomes

$$Ve = \frac{3}{5} m_0 c^2 [(1 - \beta^2)^{-1/2} (1 + \frac{1}{3}\beta^2) - 1]. \quad (15)$$

Placing Walker's expression for  $m_t$  in equation (5) gives

$$Ve = \frac{4d^2 c^2 \beta^2}{kR^2} m_0 (1 + \frac{1}{6}\beta^2)(1 - \beta^2)^{-1/2}. \quad (16)$$

Equating right members of (15) and (16) we obtain

$$R^2 = \frac{20d^2 \beta^2 (1 + \frac{1}{6}\beta^2)(1 - \beta^2)^{-1/2}}{3k[(1 - \beta^2)^{-1/2}(1 + \frac{1}{3}\beta^2) - 1]}. \quad (17)$$

Using Walker's expression for  $m_t$  and the above value of  $R^2$  in equation (8) we get

$$y = \frac{10H \frac{e}{m_0} d^2 \beta}{3ek[(1 - \beta^2)^{-1/2}(1 + \frac{1}{3}\beta^2) - 1]}. \quad (18)$$

These values of  $R$  and  $y$  are listed in columns 5 and 9 of Table I. The corresponding curve is indicated by "W" in fig. 2.

*The Bucherer-Langevin Theory.*

Assuming constant volume of the deformed electron, the expressions for the two masses of the electron as developed by Bucherer-Langevin‡ are :

$$m_t = m_0(1 - \beta^2)^{-1/3},$$

$$m_l = m_0(1 - \beta^2)^{-4/3}(1 - \frac{1}{3}\beta^2).$$

\* Walker, Proc. Roy. Soc. A 93, p. 448.

† Schott, Proc. Roy. Soc. A 94, p. 442.

‡ Bucherer-Langevin, *Phys. Zeit.* vii, p. 302.



Substituting this value of  $m_l$  in equation (1) we have

$$Ve = \int_0^\beta m_0 c^2 (1 - \beta^2)^{-4/3} (1 - \frac{1}{3}\beta^2) \beta d\beta. \quad (19)$$

Integrating by parts this becomes

$$Ve = \frac{3}{4} c^2 (m_t - m_0) + \frac{1}{4} \beta^2 c^2 m_t. \quad (20)$$

Substituting the Bucherer-Langevin expression for  $m_t$  in equation (5) and combining with (20) we obtain

$$R^2 = \frac{16d^2 \beta^2}{k[3 - 3(1 - \beta^2)^{1/3} + \beta^2]}. \quad (21)$$

Substitution in (8) now gives

$$y = \frac{H \frac{e}{m_0} R^2 (1 - \beta^2)^{1/3}}{2c\beta}. \quad (22)$$

The values of  $R$  and  $y$  calculated by the use of (21) and (22) are placed in columns 4 and 8 of Table I. The corresponding curve, "B-L," is shown in fig. 2.

### Abraham's Theory.

Assuming a rigid spherical electron Abraham\* has derived the following:

$$m_t = \frac{3}{4} \frac{m_0}{\beta^2} \left[ \frac{1 + \beta^2}{2\beta} \log_e \frac{1 + \beta}{1 - \beta} - 1 \right],$$

$$m_l = \frac{3}{4} \frac{m_0}{\beta^2} \left[ -\frac{1}{\beta} \log_e \frac{1 + \beta}{1 - \beta} + \frac{2}{1 - \beta^2} \right].$$

Substituting his value of  $m_l$  in equation (1) we have

$$Ve = \frac{3}{4} m_0 c^2 \int_0^\beta \frac{1}{\beta} \left[ -\frac{1}{\beta} \log_e \frac{1 + \beta}{1 - \beta} + \frac{2}{1 - \beta^2} \right] d\beta,$$

$$Ve = \frac{3}{4} m_0 c^2 \left\{ \int_0^\beta -\frac{1}{\beta^2} \log_e \frac{1 + \beta}{1 - \beta} d\beta + \int_0^\beta \frac{2}{\beta(1 - \beta^2)} d\beta \right\}. \quad (23)$$

On integrating the first term by parts the equation simplifies readily and becomes

$$Ve = \left[ \frac{3m_0 c^2}{4\beta} \log_e \frac{1 + \beta}{1 - \beta} \right]_0^\beta. \quad (24)$$

\* Abraham, *Ann. d. Phys.* x. p. 105.

Expanding in series and substituting limits

$$Ve = \frac{3m_0c^2}{2} \left[ \frac{1}{2\beta} \log_e \frac{1+\beta}{1-\beta} - 1 \right]. \quad (25)$$

Placing Abraham's expression for  $m_t$  in equation (5) we have

$$Ve = \frac{3m_0c^2d^2}{2kR^2} \left[ \frac{1+\beta^2}{\beta} \log_e \frac{1+\beta}{1-\beta} - 2 \right]. \quad (26)$$

Equating right members of (25) and (26) we get

$$R^2 = \frac{\frac{d^2}{k} \left[ \frac{1+\beta^2}{\beta} \log_e \frac{1+\beta}{1-\beta} - 2 \right]}{\frac{1}{2\beta} \log_e \frac{1+\beta}{1-\beta} - 1}. \quad (27)$$

Substituting Abraham's value of  $m_t$  and the above value of  $R^2$  in equation (8) we obtain

$$y = \frac{2H \frac{e}{m_0} R^2 \beta}{3e \left[ \frac{1+\beta^2}{2\beta} \log_e \frac{1+\beta}{1-\beta} - 1 \right]}. \quad (28)$$

These values of  $R$  and  $y$  are listed in columns 3 and 7 of Table I. The curve is designated by "A" in fig. 2.

The maximum voltage applied cannot exceed  $2 \times 10^5$  volts. The curves are plotted as solid lines for velocities up to  $\beta = .7$ , as this is the region that will be covered experimentally. The curves are continued as dotted lines up to  $\beta = .9$ .

The terminal points of the curves indicate that the application of a constant voltage and zero magnetic field could also be used as a test. The resulting trace on the photographic film would then be a single spot, the distance of the spot from the source enabling the experimenter to determine which of the theories is most nearly correct. It is readily seen, however, that this test becomes the more delicate only in the region  $\beta = .9$ . This region cannot be explored at the present time.

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LXXV. *Does an accelerated Electron necessarily radiate Energy on the Classical Theory?* By G. A. SCHOTT, B.A., D.Sc., F.R.S., Professor of Applied Mathematics, University College of Wales, Aberystwyth\*.

1. **I**N a paper † and a letter ‡, published under the above title, S. R. Milner has referred to a solution of mine of the problem of the motion of a Lorentz electron in a uniform electric field, when the motion is parallel to the lines of force of the field. It is a special case of the more general problem of the monoclinoidal motion of an electron solved in my Adams Prize Essay of 1908, but treated more completely in my book on Electromagnetic Radiation in §§151–154, as regards the motion itself, and §§43–60, as regards the electromagnetic field produced by it. In my book (quoted below as E.R.) no account was taken of the effect of the reaction due to radiation, beyond an estimate of the numerical error in E.R. § 154; but in a later paper in this journal § I worked out the effect of the radiation pressure completely for the special case considered by Milner. The result was unexpected: the radiation pressure was found to be zero, provided that Newton's First and Second Laws of Motion were assumed to be true when the velocity of the electron was infinitely small compared with that of light. The explanation is simple; there is an irreversible radiation of energy at the rate  $R$  given by the well-known formula of Liénard and Abraham, but this is not all. There is also a reversible radiation at the rate  $-\dot{Q}$ , where  $Q = 2ce^2(\nabla\mathbf{v})/3(c^2 - v^2)^2$ ;  $-\dot{Q}$ , being a function of the motion of the electron alone and representing energy stored in it, may be called its acceleration energy. Calculation shows that  $R - \dot{Q}$  is identically zero, so that the whole of the energy radiated irreversibly between any two times is derived from the acceleration energy, whilst all the work done by the electric field goes to increase the kinetic energy of the electron. Thus on the whole no energy is radiated in any interval whatsoever.

2. This result is in complete agreement with one of the

\* Communicated by the Author.

† S. R. Milner, *Phil. Mag.* vol. xli. p. 405 (1921) (quoted below as M1).

‡ S. R. Milner, *Phil. Mag.* vol. xliv. p. 1052 (1922) (quoted below as M2).

§ G. A. Schott, *Phil. Mag.* vol. xxix. p. 49 (1915) (quoted below as S).  
*Phil. Mag.* S. 6. Vol. 45. No. 268. April 1923 3 D

conclusions drawn by Born \* in his 1909 paper on the Theory of the rigid Electron in the Kinematics of the Principle of Relativity, referred to by Milner in his letter (M2) ; but the point of view of Born is so different from mine that at the time I did not realize the connexion between his investigation and mine, otherwise I should have referred to his paper in my book as well as my last paper (S). Nevertheless there are some points of difference, to which it is worth while to draw attention, due in the main to the fact that Born takes the standpoint of an observer moving with the electron, whilst mine is that of a fixed observer, who studies the motion of the electron relative to himself. Consequently Born finds that the components of the 4-vector potential of the Special Relativity Theory for the fixed observer can be expressed in terms of those of a "resting" 4-vector potential, relative to the moving observer, by means of a linear transformation of the Lorentz type, when the motion is that of our special case, identical with Born's "hyperbolic" motion. Moreover the "resting" 4-vector potential depends only on the three parameters of the world line of the fieldpoint under consideration and not at all on the fourth parameter, which determines the position of the field point on its world line ; Born concludes that the electron carries its electromagnetic field with it. He also deduces the result that the magnetic force, and therefore also the Poynting vector, vanishes for the moving observer, so that to him the electron does not appear to radiate. This result is hardly surprising in view of the fact that the electron always remains at rest and invariable in form for this observer. But I can find no proof in Born's paper, nor have I been able to find one anywhere else, that the vanishing of the Poynting vector relative to an observer moving with an electron which executes a "hyperbolic" motion necessarily entails the cancelling of the irreversible by the reversible radiation for a fixed observer ; thus the formal proof that this does take place, given in (S), seems to be necessary.

3. Again in §13 of his paper Born proves that the term of zero order in the dimensions of the electron occurring in the expression for the mechanical force, *i. e.*, what I have called the radiation pressure, vanishes identically. This term involves Born's coefficient  $\mu_1$  as a factor, but in evaluating it the assumption is made that the electron appears to an observer moving with it to possess spherical symmetry. This limitation is unnecessary, for the radiation pressure is

\* Born, *Annalen der Physik*, iii. p. 1 (1909).

independent of the structure of the electron, a fact already known to Abraham.

4. Returning now to Milner's first paper (M 1) we note that his interesting modification of the problem considered so far is one in which two electrons, equal in every respect but with charges of opposite sign, are moving in the same straight line with equal but opposite velocities and accelerations. In this modification there is no radiation because of the quasisymmetry of the electromagnetic field, due to the annihilation of the boundary  $x+ct=0$ , but this proves nothing as to the radiation from a single accelerated electron. The absence of radiation in Milner's case seems to me somewhat analogous to the reduction of the radiation from a revolving ring of equidistant electrons owing to the superposition of the electromagnetic fields of the different electrons of the ring; but whereas in Milner's case the superposition of the fields annihilates the radiation completely, it does so only partially in the case of the ring.

5. In the same paper (M 1, p. 417) Milner raises a fundamental question as to the meaning of the undoubted fact that there is on the whole a flow of energy from the moving boundary  $x+ct=0$  into the electromagnetic field of the accelerated electron, and concludes that "the solution premises an initial intrinsic energy in the boundary, apart from that of the electronic field." This conclusion does not appear to me to be warranted, for there must be an electromagnetic field beyond the moving boundary, due to the past history of the accelerated electron; I shall now prove that whatever that past history may have been, provided only that the velocity of the electron has been a continuous function of the time, the electromagnetic field beyond the moving boundary supplies the boundary with precisely the amount of energy needed to compensate the flow of energy into the electronic field. Let us begin by calculating the flow in question.

The line of electric force due to the electron,  $\chi=\text{constant}$ , is a circle of radius  $\xi \operatorname{cosec} \chi$ , of which the centre is the point  $(0, \xi \cot \chi)$ , and the line of the Poynting flux is the orthogonal circle. Hence the sine of the angle made by the line of force with the axis of  $x$  is numerically equal to  $x \sin \chi / \xi$ , which reduces to  $ct \sin \chi / \xi = \beta \sin \chi$  at the moving boundary. It follows from Milner's equation (4), (M 1, p. 406) that at the moving boundary  $E_y = \beta \sin \chi \cdot E = H$ . The rate at which the moving boundary lays down energy behind it is  $c(E^2 + H^2)/8\pi$ , when there is no external field, or  $c(\{E_x + X\}^2 + E_y^2 + H^2)/8\pi$  when there is. The latter



expression reduces to  $c(\{E_x + X\}^2 + 2H^2)/8\pi$  in virtue of the relation just proved. The component of the Poynting flux normal to the boundary is equal to  $cE_y H/4\pi = cH^2/4\pi$  by the same relation, and it is from the electronic field towards the boundary. Hence the total rate at which the electronic field gains energy from the boundary is the difference between these two quantities and is equal to  $c\{E_x + X\}^2/8\pi$ .

But  $E_x + X$  is the flux of electric induction from the boundary into the electronic field reckoned per unit area. The boundary represents an infinitely thin transition layer; the electric and magnetic forces in it are everywhere finite, and it is uncharged. Therefore there can be no flux of induction along it, and there must be a flux of induction into it from the region outside the electronic field exactly compensating the first flux,  $E_x + X$ . Thus there must be a field outside, such that the normal electric force is the same on both sides of the moving boundary. This outside field is that due to the motion of the electron which existed before the establishment of the external electric field  $X$  producing the accelerated motion, and like all fields due to the motion of a single electron it possesses the property that the component of the electric force tangential to its boundary is exactly equal to the magnetic force, a result which follows from the usual point-law (*cf.* M 1, p. 416, or E. R., p. 23, equations 32 and 33). Hence there is on the whole a loss of energy from the outside field to the moving boundary equal to  $c\{E_x + X\}^2/8\pi$ , per second, precisely equal to the gain of the inside electronic field. It should be noted that if the velocity of the electron be suddenly changed as well as its acceleration, this is no longer true; for then an X-ray pulse is generated, inside which the electric and magnetic forces are very intense, so that there is a finite flux of induction along it, and the normal electric forces on the two sides of the pulse are no longer equal.

6. In conclusion, the whole question of the radiation from a single accelerated electron will be put in a clearer light by solving the following problem:—Required to find the conditions under which an accelerated electron will not radiate energy on the whole and will move as if there were no radiation, in the usual sense of irreversible radiation. This problem is most easily attacked by the method used in the paper already referred to (S, §§ 3, 4).

For the sake of simplicity we shall use as units of mass, length, and time respectively the following quantities: the resting mass of the electron,  $m = 8.76.10^{-28}$  gram, the

conventional radius of the resting surface electron,  $2e^2/3c^2m = 1.83.10^{-13}$  cm., and, lastly,  $2e^2/3c^3m = 6.1.10^{-24}$  sec. Instead of the time  $t$  we shall use the time variable  $\tau$ , defined by the equation  $\tau = \int_0^t \sqrt{1-v^2} dt$ , where  $v$  is the velocity of the electron referred to that of light as the appropriate new unit. We shall use an accent to denote differentiation with respect to  $\tau$ , and further shall write  $\mathbf{w} = \mathbf{r}' =$  vector velocity referred to time  $\tau$ . Then the equation of motion of the electron under force  $\mathbf{F}$  may be written in the form (S, p. 52, equations (43) and (44))

$$\mathbf{w}' - \mathbf{w}'' + R\mathbf{w} = \mathbf{F} \sqrt{1+w^2} \quad \dots \quad (1)$$

where  $R$  is the irreversible radiation and is given by

$$R = \mathbf{w}'^2 - \frac{(\mathbf{w}\mathbf{w}')^2}{1+w^2} \quad \dots \quad (2)$$

The first term on the left of (1) represents the usual effective force on the electron, whilst the last two represent the radiation pressure; when the radiation has no effect on the motion, the last terms cancel each other, and the equation breaks up into two:—

$$\mathbf{w}'' = \left\{ \mathbf{w}'^2 - \frac{(\mathbf{w}\mathbf{w}')^2}{1+w^2} \right\} \mathbf{w} \quad \dots \quad (3)$$

and  $\mathbf{w}' = \mathbf{F} \sqrt{1+w^2} \quad \dots \quad (4)$

Multiplying (3) vectorially by  $\mathbf{w}$  we obtain in succession

$$[\mathbf{w}\mathbf{w}''] = 0, \quad [\mathbf{w}\mathbf{w}'] = \mathbf{C}, \quad (\mathbf{C}\mathbf{w}) = 0, \quad (\mathbf{C}\mathbf{r}) = D,$$

where  $\mathbf{C}$  is a vector constant, and  $D$  a scalar constant. The last equation is that of a plane, so that the orbit of the electron must be a plane curve.

7. Let us take the plane of the orbit as the  $x, y$  plane, and let  $\psi$  denote the angle between the forward direction of motion and the axis of  $x$ ; then we have  $x' = w \cos \psi$ ,  $y' = w \sin \psi$ , so that  $x', y', w, \psi$  take the place of  $x, y, r, \theta$  in plane polar coordinates. The second equation of the last set takes the place of the equation of angular momentum and may be written in the form

$$x'y'' - x''y' = w^2\psi' = C \quad \dots \quad (5)$$

The radial component of  $\mathbf{w}''$  is  $w'' - \psi'^2 w$ , the value of  $\mathbf{w}'^2$  is  $w'^2 + w^2\psi'^2$ , and that of  $(\mathbf{w}\mathbf{w}')$  is  $ww'$ . Hence the tangential component of the vector equation (3) reduces to

$$w' - w\psi'^2 = \left\{ w'^2 + w^2\psi'^2 - \frac{w^2w'^2}{1+w^2} \right\} w.$$

Using (5) we may write this in the form

$$w'' - \frac{ww'^2}{1+w^2} - C^2 \frac{1+w^2}{w^3} = 0.$$

An integrating factor of this equation is given by  $2w'/(1+w^2)$ ; performing the integration we obtain

$$\frac{w'^2}{1+w^2} + \frac{C^2}{w^2} = A. \quad (6)$$

At this stage it will be convenient to introduce new arbitrary constants in place of  $A$  and  $C$ ; we shall write

$$A = a^{-2}, \quad C = -ba^{-2}. \quad (7)$$

Then equations (5) and (6) may be written in the forms

$$w^2 \psi' = -ba^{-2}, \quad (8)$$

$$a^4 w^2 w'^2 = (1+w^2)(a^2 w^2 - b^2). \quad (9)$$

Writing  $\tau = a\phi$  for shortness' sake we obtain from (9)

$$a^2 w^2 = b^2 \cosh^2 \phi + a^2 \sinh^2 \phi. \quad (10)$$

Substituting in (8) and integrating we obtain

$$\psi = - \int^\phi \frac{ab d\phi}{b^2 \cosh^2 \phi + a^2 \sinh^2 \phi} = \tan^{-1}(b \coth \phi/a), \quad (11)$$

where the constants of integration have been chosen so as to make  $w, \psi$  equal to  $b/a, \pi/2$  respectively when  $\phi$  and  $\tau$  vanish, a choice which merely amounts to a suitable determination of the time origin and the axes, and of course involves no loss of generality.

The time  $t$  is given in terms of  $\phi$  by the equation

$$t = \int_0^\tau \sqrt{1+w^2} d\tau = \sqrt{a^2+b^2} \int_0^\phi \cosh \phi d\phi = \sqrt{a^2+b^2} \sinh \phi. \quad (12)$$

At this stage it will be convenient to change back to  $t$  as independent variable; we have

$$\dot{x} = x'/\sqrt{1+w^2} = w \cos \psi / \sqrt{1+w^2},$$

$$\dot{y} = w \sin \psi / (1+w^2).$$

From (10) and (11) we find

$$w \cos \psi = \sinh \phi, \quad w \sin \psi = b \cosh \phi/a,$$

$$\sqrt{1+w^2} = \sqrt{a^2+b^2} \cosh \phi/a.$$

From (12) we obtain

$$\dot{x} = \frac{at}{\sqrt{(a^2+b^2)}\sqrt{(a^2+b^2+t^2)}}, \quad \dot{y} = \frac{b}{\sqrt{(a^2+b^2)}}. \quad (13)$$

Integrating equations (13) and choosing the constants of integration suitably we obtain

$$x = \frac{a\sqrt{(a^2 + b^2 + t^2)}}{\sqrt{(a^2 + b^2)}}, \quad y = \frac{bt}{\sqrt{(a^2 + b^2)}} \dots (14)$$

Eliminating  $t$  we find that the orbit is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

described with a constant component velocity parallel to the transverse axis equal to  $b/\sqrt{(a^2 + b^2)}$ :

8. The resultant velocity,  $v$ , is given from (13) by

$$v = \frac{\sqrt{(b^2 + t^2)}}{\sqrt{(a^2 + b^2 + t^2)}} \dots (15)$$

Thus we obtain

$$\sqrt{(1 - v^2)} = \frac{a}{\sqrt{(a^2 + b^2 + t^2)}} \dots (16)$$

The components of the momentum in the present units are found from (13) and (16) to be

$$\frac{\dot{x}}{\sqrt{(1 - v^2)}} = \frac{t}{\sqrt{(a^2 + b^2)}}, \quad \frac{\dot{y}}{\sqrt{(1 - v^2)}} = \frac{b\sqrt{(a^2 + b^2 + t^2)}}{a\sqrt{(a^2 + b^2)}}.$$

Hence the components of the mechanical force on the electron are

$$X = \frac{1}{\sqrt{(a^2 + b^2)}}, \quad Y = \frac{bt}{a\sqrt{(a^2 + b^2)}\sqrt{(a^2 + b^2 + t^2)}} = \frac{y}{x} X \dots (17)$$

Thus the mechanical force needed to generate the motion is central with a constant component parallel to the real axis of the hyperbolic orbit. It is clearly not derivable from a potential, so that it cannot be produced by an electrostatic field alone, but it can be produced by a uniform electrostatic field of intensity  $E$  parallel to the axis of  $x$ , combined with a uniform magnetic field of intensity  $H$  parallel to the negative direction of the axis of  $z$ , *i. e.* perpendicular to the plane of the orbit. In order to see this let us for simplicity sake take the charge of the electron as unit charge; then we have by (13) and (14)

$$X = E - \dot{y}H = E - \frac{bH}{\sqrt{(a^2 + b^2)}},$$

$$Y = \dot{x}H = \frac{atH}{\sqrt{(a^2 + b^2)}\sqrt{(a^2 + b^2 + t^2)}}$$

$$\text{or} \quad Y = \frac{a^2Hy}{bx} X.$$

Comparing these equations with (17) we obtain

$$H = ba^{-2}, \quad E = a^{-2} \sqrt{(a^2 + b^2)} \dots \dots \dots (18)$$

We must bear in mind that the units are not the usual ones;  $a, b$  are lengths measured in terms of the unit  $2e^2/3c^2m$ ;  $X, Y$  are mechanical forces measured in terms of the unit  $3c^4m^2/2e^2$ ; finally,  $E, H$  are measured in terms of units which are  $3c^4m^2/2e^3$  times the electrostatic and magnetic units respectively. This factor is numerically equal to  $9.10^{15}$ . If  $a', b', H', E'$  be the values of the semiaxes of the orbit and of the magnetic and electric forces measured in centimetres and in magnetic and electrostatic units respectively, then  $a', b'$  are  $2e^2/3c^2m$  times  $a, b$  and  $H', E'$  are  $3c^4m^2/2e^3$  times  $H, E$  respectively, so that  $H' = c^2mb'/ea'^2$ , with a similar expression for  $E'$ . When  $b$  is zero, the mechanical force reduces to  $eE' = c^2m/a'$ , the value for the special case considered by Milner (M1, p. 406 (2)), where  $m_0$  is used instead of our  $m$ , and  $k$  instead of our  $a'$ .

9. It remains for us to consider the energy relations of the motion. The kinetic energy  $T$  is given by the usual relation together with (16); we obtain

$$T = \frac{1}{\sqrt{(1-v^2)}} - 1 = \frac{\sqrt{(a^2 + b^2 + t^2)}}{a} - 1, \quad \dot{T} = \frac{t}{a\sqrt{(a^2 + b^2 + t^2)}} \dots \dots \dots (19)$$

The rate of working of the applied force is found from (13) and (17) to be given by

$$X\dot{x} + Y\dot{y} = \frac{t}{a\sqrt{(a^2 + b^2 + t^2)}} = \dot{T}.$$

Thus the whole work done on the electron by the external field is transformed into kinetic energy, just as if there were no radiation. Again the rate at which the electron radiates energy irreversibly is given by the usual expression of Liénard (S, p. 51 (7)), which in our units reduces to

$$R = \frac{\dot{\mathbf{v}}^2}{(1-v^2)^2} + \frac{(\mathbf{v}\dot{\mathbf{v}})^2}{(1-v^2)^3} \dots \dots \dots (20)$$

We see from (13) that  $\ddot{x} = a\sqrt{(a^2 + b^2)}/(a^2 + b^2 + t^2)^{3/2}$ ,  $\ddot{y} = 0$ , so that  $\dot{\mathbf{v}}^2 = \dot{x}^2$ , and  $(\mathbf{v}\dot{\mathbf{v}}) = \dot{x}\ddot{x}$ ; hence we obtain from (20), with (16),

$$R = \frac{(1-\dot{y}^2)\dot{x}^2}{(1-v^2)^3} = \frac{1}{a^2} \dots \dots \dots (21)$$

Thus the electron radiates energy irreversibly at a constant rate.



Lastly, the acceleration energy,  $-Q$ , is given in our units by the following equation (*cf.* S, p. 51 (6))

$$Q = \frac{(\dot{\mathbf{v}}\dot{\mathbf{v}})}{(1-v^2)^2} = \frac{\ddot{x}\ddot{x}}{(1-v^2)^2} = \frac{t}{a^2} \dots \dots (22)$$

Comparing this equation with (21) we see that  $R - \dot{Q} = 0$ .

Thus the irreversible loss of energy by radiation is exactly defrayed from the loss of acceleration energy of the electron, just as it was found to be in the special case of the hyperbolic motion (S, § 8) already referred to.

The accelerated motions just considered are particularly interesting on account of this property of emitting no radiation on the whole, and we have proved that they are unique in this respect. It must be noted particularly that they do not help us in any way to remove the existing contradiction between the invariable irreversible loss of energy, due to radiation from an electron describing a closed orbit, required by classical electrodynamics, and the total absence of radiation in stationary motions of electrons postulated by Bohr. It is otherwise evident that the acceleration energy of an electron describing a stationary orbit, depending as it does on the state of the electron alone, must repeat its values as that state repeats itself, and cannot therefore provide a reservoir from which the irreversible part of the radiation can be defrayed permanently. On this question reference may be made to a former communication\* and to a paper by H. Bateman †, in which the writer's conclusions are confirmed.

10. The chief conclusions of the present paper may be summarized as follows:—

(1) The hyperbolic motion of the electron considered in (S) and (M1) involves no contradiction with the Principle of the Conservation of Energy provided that it be generated in any way which is consistent with the conservation of its electric charge and the continuity of its velocity.

(2) The only class of motions unaffected by radiation is that in which the path of the electron is an hyperbola described with a constant component velocity parallel to the conjugate axis. The hyperbolic motion of Born is a special case in which the transverse velocity is zero. In these motions the radiation is defrayed entirely at the expense of the acceleration energy of the electron.

\* Schott, *Phil. Mag.* vol. xxxvi. p. 243 (1918).

† Bateman, *Nat. Acad. Sci., Proc.* 5, p. 367 (1919).

LXXVI. *The Disturbance of the Uniform Temperature of the Stratosphere by the Vertical Displacement associated with Horizontal Motion governed by the "Geostrophic Law."*  
By F. J. W. WHIPPLE, M.A., F.Inst.P.\*

§ 1. **I**N his work on 'Weather Prediction by Numerical Process' †, Mr. L. F. Richardson has devoted much attention to the results which follow from the assumption that the flow of air is governed by the law that the "geostrophic force" due to the earth's rotation balances exactly the pressure gradient.

The analysis appropriate for the general motion of the atmosphere is necessarily complicated, and therefore special interest is attached to a very neat formula given ‡ by Mr. Richardson, by which certain processes taking place in the stratosphere are related.

The formula in question is such that it can be tested by the comparison of the records of wind velocity and of air temperature. A preliminary test was made in Mr. Richardson's book, and many more are given in a recently published paper § entitled "An observational test of the geostrophic approximation in the Stratosphere," in which he collaborated with Drs. Wagner and Dietzius.

The following investigation, which was undertaken with the intention of supplying a simple proof of Richardson's formula, has led to the discovery that the formula is incomplete and that the term which had been overlooked is of considerable importance.

§ 2. The problem to be investigated may be set out as follows:—It being assumed (1) that the horizontal movement of the air is at all times given precisely by the "geostrophic law" ||, (2) that the distribution of temperature

\* Communicated by the Author.

† Cambridge University Press, 1922.

‡ *L. c.* page 144, equation 18.

§ London R. Met. Soc. Q. J. 1922, p. 328.

|| Sir Napier Shaw writes as follows ('Manual of Meteorology,' part iv. p. 4: Cambridge University Press, 1919):—"The function of pressure-distribution seems rather to be to steer the air than to speed it or stop it. So it will be more profitable to consider the 'strophic' balance between the flow of air and the distribution of pressure as an axiom or principle of atmospheric motion. . . . This principle was enunciated in a paper before the Royal Society of Edinburgh in 1913 as follows:—*In the upper layers of the atmosphere the steady horizontal motion of the air at any level is along the horizontal section of the isobaric surface at that level and the velocity is inversely proportional to the separation of the isobaric lines in the level of the section.*"

in the stratosphere at a particular instant is independent of height though not uniform in a horizontal plane, and (3) that the effects of radiation and eddy conductivity may be ignored; what will be the initial rate of change of temperature?

To solve the problem we have to express in mathematical form the laws of conservation of mass and conservation of energy.

The conservation of mass implies that the rate of increase of density in an elementary volume moving with the air is accounted for by the excess of inflow over outflow. As regards vertical movement, the excess is  $-\rho \frac{\partial V_H}{\partial h}$  ( $\rho$  being the density,  $V_H$  the upward velocity, and  $h$  being measured upwards). As regards horizontal movement we notice that when the dependence of the geostrophic factor  $2\omega \sin \phi$  on latitude is ignored, the assumption that the horizontal velocity is "geostrophic" implies stream-line motion with no accumulation of matter anywhere. To make allowance for the latitude variation we need consider only the northward component of velocity, for which the equation

$$V_N = \frac{1}{2\omega \sin \phi} \cdot \frac{1}{\rho} \cdot \frac{\partial \rho}{\partial e} \dots \dots (1)$$

holds. ( $\omega$  is the angular velocity of the earth,  $\phi$  the latitude, and  $\frac{\partial \rho}{\partial e}$  is the east-gradient of pressure.)

The increase of  $V_N$  towards the north on account of the variation of  $\phi$  is given by the equation

$$\frac{1}{V_N} \cdot \frac{\partial V_N}{\partial n} = - \frac{1}{\sin \phi} \cdot \frac{\partial \sin \phi}{\partial n} = - \frac{\cot \phi}{a} \dots (2)$$

( $a$  is the radius of the earth,  $dn = a d\phi$ , and the notation  $\frac{\partial V_N}{\partial n}$  is used to indicate that only the variation of  $V_N$  dependent on the variation of  $\sin \phi$  is considered).

Accordingly the equation representing the conservation of mass is

$$\frac{D\rho}{Dt} = -\rho \left[ \frac{\partial V_H}{\partial h} - V_N \frac{\cot \phi}{a} \right] \dots \dots (3)$$

By hypothesis the effects of radiation etc. are excluded, so that the conservation of energy is represented by the

780 Mr. F. J. W. Whipple on the Disturbance of the  
adiabatic conditions :

$$\frac{1}{\gamma_v} \frac{D}{Dt} (\log \rho) = \frac{1}{b} \frac{D}{Dt} (\log \theta) = \frac{1}{\gamma_p} \frac{D}{Dt} (\log p), \quad (4)$$

in which  $\gamma_v$ ,  $\gamma_p$  are the capacities for heat of air at constant volume and constant pressure (in dynamical measure) and  $b$  is the "gas constant," so that  $\gamma_p - \gamma_v = b$ .

Now

$$\frac{D}{Dt} (\log \theta) = \left( \frac{\partial}{\partial t} + V_H \frac{\partial}{\partial h} \right) \log \theta + \left( V_E \frac{\partial}{\partial e} + V_N \frac{\partial}{\partial n} \right) \log \theta, \quad (5)$$

and since the horizontal wind is geostrophic

$$V_N = \frac{1}{2\omega \sin \phi} \cdot \frac{1}{\rho} \cdot \frac{\partial \rho}{\partial e} = \frac{\theta \cdot b}{2\omega \sin \phi} \frac{\partial}{\partial e} (\log p), \quad (6)$$

whilst

$$V_E = -\frac{1}{2\omega \sin \phi} \cdot \frac{1}{\rho} \frac{\partial \rho}{\partial n} = -\frac{\theta \cdot b}{2\omega \sin \phi} \frac{\partial}{\partial n} (\log p). \quad (7)$$

It follows that

$$\frac{1}{b} \frac{D}{Dt} (\log \theta) = \frac{1}{b} \left( \frac{\partial}{\partial t} + V_H \frac{\partial}{\partial h} \right) \log \theta + \frac{1}{2\omega \sin \phi} \frac{\partial (\log p, \theta)}{\partial (e, n)}. \quad (8)$$

In writing down the corresponding equation for variation of  $p$ , we notice (1) that the horizontal motion, being along the isobars, does not necessitate any change of pressure, and (2) that to a close approximation, vertical accelerations being small in comparison with  $g$ ,

$$\frac{\partial}{\partial h} (\log p) = -\frac{g}{b\theta} \quad (9)$$

Accordingly we have

$$\frac{D}{Dt} (\log p) = \frac{\partial}{\partial t} (\log p) - V_H \frac{g}{b\theta} \quad (10)$$

§ 3. The foregoing equations do not involve the assumption as to the uniformity of temperature in the stratosphere.

We now follow Richardson and suppose that  $\frac{\partial \theta}{\partial h} = 0$  holds instantaneously, but that  $\frac{\partial \theta}{\partial t}$  and its derivatives do not vanish.

Under these conditions differentiation of the formulæ leads to simplification.

The relations we propose to utilise are derived from (4), viz. :—

$$\frac{1}{\gamma_v} \frac{\partial}{\partial h} \frac{D}{Dt} (\log \rho) = \frac{1}{b} \frac{\partial}{\partial h} \frac{D}{Dt} (\log \theta) \quad . \quad . \quad (11)$$

and

$$\frac{1}{\gamma_p} \frac{\partial^2}{\partial h^2} \frac{D}{Dt} (\log p) = \frac{1}{b} \frac{\partial^2}{\partial h^2} \frac{D}{Dt} (\log \theta). \quad . \quad . \quad (12)$$

On substitution of the appropriate terms from (3), (8), and (10) these relations become

$$-\frac{1}{\gamma_v} \left[ \frac{\partial^2 V_H}{\partial h^2} - \frac{\partial V_N}{\partial h} \frac{\cot \phi}{a} \right] = \frac{1}{b} \frac{\partial}{\partial h} \frac{\partial}{\partial t} (\log \theta) \quad (13)$$

and

$$\frac{1}{\gamma_p} \frac{g}{b\theta} \left[ \frac{\partial}{\partial h} \frac{\partial}{\partial t} \log \theta - \frac{\partial^2 V_H}{\partial h^2} \right] = \frac{1}{b} \frac{\partial^2}{\partial h^2} \frac{\partial}{\partial t} \log \theta. \quad (14)$$

Proceeding to eliminate  $\frac{\partial^2 V_H}{\partial h^2}$  we obtain the relation

$$\frac{b}{\gamma_p} \frac{\cot \phi}{a} \frac{\partial V_N}{\partial h} = \frac{\partial}{\partial h} \frac{\partial}{\partial t} \log \theta - H \frac{\partial^2}{\partial h^2} \frac{\partial}{\partial t} \log \theta, \quad . \quad (15)$$

where  $H$  is written for  $\frac{b\theta}{g}$  or  $\frac{p}{g\rho}$  the height of the “homogeneous atmosphere” at the temperature  $\theta$ .

The term in  $H$  is the one which was overlooked \* by Mr. Richardson.

§ 4. The interest in this formula (15) appears to be mainly theoretical in that it shows one way in which temperature differences would arise in the stratosphere, even if there were complete uniformity to begin with. An observational test is hardly a fair one unless due allowance is made for any departures from uniformity that may be known.

\* The error can be traced to the omission of a term  $\gamma_v \frac{\partial}{\partial t} \left( \frac{\partial \log \theta}{\partial h} \right)$  from the left of equation (10) on p. 142 of ‘Weather Prediction by Numerical Process.’ In passing from (9) to (10) it is not legitimate to use (7) and differentiate with respect to the time, for (7) is proved on the assumption that  $\frac{\partial \theta}{\partial h} = 0$ , and is therefore only valid as a statement of the initial conditions. The disappearance of the term  $\tan \phi$  from Mr. Richardson’s equation (10) is in accordance with the Met. Soc. paper (*l. c.* p. 329).



In arriving at (15) it was not only assumed that  $\frac{\partial \theta}{\partial h}$  was zero, but also that its differential coefficients with respect to  $h$  and  $e$  and  $n$  were zero. If these assumptions are not made then (15) must be replaced by a much more complicated relation: and the terms between which comparison has to be made are not only  $\frac{\partial}{\partial h} \frac{\partial}{\partial t} \log \theta$  and  $H \frac{\partial^2}{\partial h^2} \frac{\partial}{\partial t} \log \theta$ , but also such as

$$V_H \frac{\partial^2}{\partial h^2} \log \theta, \quad H V_H \frac{\partial^3 \log \theta}{\partial h^3}, \quad V_N \frac{\partial^2 \log \theta}{\partial n \partial h}, \quad \text{and} \quad V_N H \frac{\partial^3 \log \theta}{\partial n \partial h^2}.$$

In the examples collected by Richardson, Wagner, and Dietzius,  $\frac{\partial}{\partial h} (\log \theta)$  is not usually zero, and there is no reason to suppose that these terms, which represent the irregular space distribution of  $\frac{\partial}{\partial h} (\log \theta)$ , are small in comparison with the terms which represent the rate at which it is changing.

In the circumstances the coefficient computed by these authors for the correlation between the term on the left of equation (15) and the first term on the right is higher than one would expect.

§ 5. In conclusion, it may be remarked that the general equations (4), in which  $V_H$  and  $\frac{\partial V_H}{\partial h}$  occur implicitly, as may be seen by comparison with equations (3), (8), and (10), provide opportunities for comparison of theory and observation. Observations of  $V_H$  up to very great heights by the pilot-balloon method are not numerous, but they should serve for the purpose. It will be noticed that the equations mentioned hold good for the troposphere, and therefore that the investigation can refer to the region in which the observations of temperature are being made day by day instead of to the higher layers which are only reached occasionally by balloons.

LXXVII. *Note on the Calculation of High-Frequency Inductances.* By F. B. PIDDUCK \*.

THE present note has been suggested by certain experiments †, in which the self-inductance of a circular ring of rectangular cross-section was measured for oscillatory currents of medium frequency. A rough estimate of the theoretical inductance can be found by taking the inductance for low frequency from Maxwell's or Weinstein's formula and correcting as for a circular cross-section of equal area. The principles on which a solution of the general problem is to be sought have been stated by Rayleigh ‡ and Curtis §. Rayleigh's method (easily generalized to any frequency and cross-section) is the more powerful, but will not give the self-inductance without prohibitive labour, unless the frequency is low. There is, however, one case of extreme simplicity, that of a thin ring (of any cross-section) at very high frequency. I propose to show how the self-inductance can be found when the distribution of static electricity on a long bar of the same cross-section is known.

The theory is based most simply on an argument of Maxwell ||, quoted by Rayleigh. Suppose that the inducing field is confined to a region which may interlace the ring, but does not cut into it, so that the impressed magnetic force vanishes at all points of the metal. The induced currents then have no magnetic force inside the metal, and a purely tangential force at its surface. The argument applies so far to any conducting mass whatever, at sufficiently high frequency. It follows that the current in a *thin* ring is spread over the surface in such a way that the surface-density of current is proportional to the surface-density of static electricity on a long bar of the same section, the lines of magnetic force near the wire corresponding to the equipotentials. The currents are all in the same phase, a quarter period behind the inducing field. This conclusion follows also from Rayleigh's theory of waves on wires ¶.

Consider for the moment a slightly more general distribution. Let  $u$  be the amplitude of current-density at a

\* Communicated by the Author.

† Phil. Mag. [6] vol. xlii. p. 220 (1921).

‡ Rayleigh, Proc. Roy. Soc. A, vol. lxxxvi. p. 567 (1912); Scientific Papers, vol. vi. p. 106.

§ H. L. Curtis, Sci. Papers Bur. Standards, vol. xvi. p. 93 (1920).

|| 'Electricity and Magnetism,' 3rd ed. vol. ii. p. 290.

¶ Rayleigh, Phil. Mag. [5] vol. xlv. p. 199 (1897); Scientific Papers, vol. iv. p. 327.

point P of the cross-section. Take another point P' distant  $r$  from P in the same section, and let  $dS'$  be an element of area at P'. Write  $W$  for the logarithmic potential of the distribution  $u$  at P; that is,

$$W = \int u' \log r dS', \quad . . . . . (1)$$

where  $u'$  is the value of  $u$  at P' and integration is over the whole section. The lines of current-flow at P and P' form two closed filaments whose mutual inductance  $M$  becomes logarithmically infinite as P' tends to P, and may be written  $B \log r + C$ , where  $B$  and  $C$  are constant for a given form of loop. The self-inductance  $L$  of the ring is twice the magnetic potential energy of the different filaments, with sign changed, for a total current unity. Thus

$$L = \iint M u u' dS dS', \quad \text{where} \quad \int u dS = 1.$$

Hence

$$L = \iint (B \log r + C) u u' dS dS' = B \int u W dS + C = B \log R + C,$$

where

$$\log R = \int u W dS. \quad . . . . . (2)$$

This is a generalization of Maxwell's method of geometrical mean distance\*, admitting of easy further generalization to the case of variable phase. The self-inductance is equal to the mutual inductance of two fictitious lines, the line of centroids of the cross-section and a curve running parallel to it at a distance  $R$  given by (2). Since  $W$  is constant,

$$\log R = W, \quad \text{with} \quad \int u dS = 1. \quad . . . . . (3)$$

Thus  $\log R$  is the true logarithmic potential of the distribution of current. It is more convenient to deal with differences of potential. Let  $V$  be the potential at a point Q outside the conductor, the conductor itself being at *zero potential*. The true logarithmic potential at Q is  $W + V$ , and this tends to  $\log s$  as Q moves to infinity, since the total current is unity,  $s$  being the distance of Q from any finite point in the plane of the section. Hence  $W$  is the limit of  $\log s - V$  as Q goes to infinity, and we are now free to use the ordinary modes of solution of the electrostatic problem in two dimensions. Using the method of conjugate functions, suppose that the space outside the cross-section (in the  $z$ -plane) is represented conformally on the outside of a circle of unit radius (in the  $\zeta$ -plane), so that the two perimeters correspond, the origin in the  $\zeta$ -plane being at the centre of

\* 'Electricity and Magnetism,' 3rd ed. vol. ii. p. 324.

the circle. The relation between  $z$  and  $\zeta$  is assumed to be known; this is what we mean by saying that the electrostatic problem has been solved. The further substitution

$$V + iU = \log \zeta$$

makes  $V=0$  when  $|\zeta|=1$ , and gives the potential at any external point when the perimeter is at zero potential. Taking the point from which  $s$  is measured as the origin in the  $z$ -plane,  $s=|z|$  and  $V=\log|\zeta|$ , so that

$$\log s - V = \log |z/\zeta|.$$

Hence  $R$  is the limit of  $|z/\zeta|$  as  $Q$  tends to infinity. The infinitely distant parts of the two planes are similar to each other, and  $|z/\zeta|$  is the ratio of the linear dimensions. The elementary lengths  $|dz|$  and  $|d\zeta|$  are in the same ratio. Hence we can write indifferently

$$R = \lim \left| \frac{z}{\zeta} \right| = \lim \left| \frac{dz}{d\zeta} \right|, \quad . . . . (4)$$

as may be most convenient. Thus, if the representation of the outside of the cross-section on the outside of a circle of unit radius changes distant linear dimensions in the ratio  $R$  to 1,  $R$  is the high-frequency geometrical mean distance of the section. Otherwise,  $R$  is the radius of that circular cylinder which forms, with a parallel cylindrical sheath of very large radius, a condenser of the same capacity per unit length as the wire itself, when straightened out.

The external representation of a rectangle on a circle of unit radius belongs to a class of problems treated by Schwarz\*. The transformation is

$$\frac{dz}{d\zeta} = -iC \frac{(\zeta^4 - 2\zeta^2 \cos 2\alpha + 1)^{\frac{1}{2}}}{\zeta^2}, \quad . . . (5)$$

where the radical is positive at the point  $\zeta=1$ , and  $C, \alpha$  are real constants. The corners of the rectangle correspond to vectorial angles  $\alpha, \pi - \alpha, \pi + \alpha$  and  $2\pi - \alpha$  on the unit circle. If the origin of  $z$  corresponds to  $\zeta=1$ , the corners are at the points  $\frac{1}{2}b, \frac{1}{2}b + ic, -\frac{1}{2}b + ic$  and  $-\frac{1}{2}b$  in the  $z$ -plane, where  $b$  and  $c$  are the lengths of the sides. The integral form of (5) then yields two equations for  $C$  and  $\alpha$  in terms of  $b$  and  $c$ , which become on reduction

$$4C(E - \cos^2 \alpha K) = b, \quad 4C(E' - \sin^2 \alpha K') = c,$$

where  $K$  and  $E$  are complete elliptic integrals of the first

\* H. A. Schwarz, *Jour. f. Math.* (Crelle), vol. lxx. p. 115 (1869); *Ges. Abhandlungen*, vol. ii. p. 77.

and second kind to modulus  $\sin \alpha$ , and  $K'$ ,  $E'$  those to modulus  $\cos \alpha$ . We find  $\alpha$  by trial to make  $b/c$  correct, and then either equation determines  $C$ . Since  $dz/d\xi$  tends to  $-iC$  at infinity,  $R=C$ .

Let us apply the result to the circular ring of mean radius  $a=1.735$  cm. and rectangular section  $0.35$  cm. by  $0.33$  cm., the inductance of which at  $3250$  metres was measured as  $52.1$  cm. We find  $\alpha=44^\circ 13'$ ,  $C=0.2005$ . The high-frequency inductance is therefore, to a first approximation,

$$L=4\pi a \left( \log_e \frac{8a}{R} - 2 \right) = 48.77 \text{ cm.}$$

The low-frequency inductance, to the same order (Maxwell's formula) being  $54.87$  cm., there is a loss of  $6.10$  cm. from low to high frequency. The loss for a circular cross-section would be half the mean perimeter, or  $5.46$  cm., so that the estimated inductance of  $51.0$  cm. at  $3250$  metres may be too high by  $0.3$  cm. or so, whereby the agreement with experiment becomes slightly worse. It is satisfactory to find that the margin of theoretical uncertainty is quite small.

LXXVIII. *Critical Electron Energies in Helium, and the Extreme Ultra-Violet Spectrum.* By ANN CATHERINE DAVIES, *D Sc., Royal Holloway College, Englefield Green*.\*.

IN a notice in 'Nature' of August 26th, 1922, Professor Lyman reported the detection of several hitherto unobserved lines in the extreme ultra-violet spectrum of helium. The lines in question were at  $584.4$ ,  $537.1$ ,  $522.3$ , and  $515.7$ , ÅU, correct to within one or two tenths of a unit, and a faint line at  $600.5 \pm 0.3$  ÅU., and they were detected by maintaining a good vacuum in the body of the vacuum spectroscop, while the discharge tube contained helium at about  $1$  mm. pressure. No separating window was used between the discharge tube and the spectroscop, but a short narrow slit separated them and a powerful pump was kept in action. Previous to this the only helium line which had been detected in this region of the spectrum was  $\lambda 585$ , now given as  $584.4$  ÅU, and this had been observed by Fricke and Lyman †.

Of the lines referred to above  $\lambda 584.4$  was found to be the most intense, and the intensities of the next three lines were

\* Communicated by Prof. F. Horton.

† H. Fricke and T. Lyman, *Phil. Mag.* xli. p. 814 (1921).



found to decrease with their wave-length in a manner analogous to that found with successive lines of a series. Lyman points out that the frequency differences of the first four lines are identical with the frequency differences of the first four lines of the parhelium principal series of single lines given in Fowler's notation as  $1S-mP$ , and he considers that the ultra-violet lines detected form part of the principal series  $0S-mP$ , and that the ionization energy for electrons in helium must correspond to the limit of this series. Indicating the normal state by  $N$ , it seems probable that the first four lines given by Lyman belong to the series  $N-mP$ , in which case the ionization voltage should correspond to the limit of this series,  $N$ , which can be calculated accurately and should be, as Lyman has shown, equal to 24.5 volts. The calculation does not, however, involve the assumption that  $N$  is identical with the state  $0S$ .

The value of the ionization voltage calculated by Lyman in this way does not agree with the value which has been determined experimentally by Horton and Davies\*, and by Franck and Knipping†, namely 25.2 volts, and the voltages corresponding to the lines observed by Lyman do not agree with the critical voltages which Franck and Knipping found intermediate to the fundamental resonance radiation voltages, and the ionization voltage. Lyman has, however, pointed out that a correction of  $-0.8$  volt applied to all Franck and Knipping's values for helium would bring them into good agreement with the spectroscopic data, though it would leave the lowest critical voltage with no corresponding line in the ultra-violet spectrum.

In the experiments of Franck and Knipping‡, and also those of Horton and Davies§, the higher critical voltages have all been determined relatively to the first critical voltage, and the accuracy of the absolute values given depends, therefore, upon the accuracy of the determination of the absolute value of this first critical point. In the case of the determination made by Professor Horton and the writer, the fact that identical mean values were yielded by two methods with entirely different sources of possible error, one of which might give too high a value, and

\* F. Horton and A. C. Davies, *Proc. Roy. Soc. A*, vol. xcv. p. 408 (1919), and *Phil. Mag.* vol. xlii. p. 746 (1921).

† J. Franck and P. Knipping, *Phys. Zeits.* vol. xx. p. 481 (1919), and *Zeits. f. Phys.* vol. i. p. 320 (1920).

‡ *Loc. cit.*

§ *Loc. cit.*, also F. Horton and A. C. Davies, *Phil. Mag.* vol. xxxix. p. 592 (1920); and A. C. Davies, *Proc. Roy. Soc. A*, vol. c. p. 599 (1922).

the other of which might give too low a value, was taken as indicating that the errors were negligible; the common mean value was therefore taken as the true absolute value of the first critical electron energy. For the details of the apparatus and the arrangements of electric fields employed in the two methods, the reader is referred to the original account of the experiments (Proc. Roy. Soc. A, vol. xcv. p. 408, 1919). Only the essential points will be given here.

The first method consisted in ascertaining the minimum value of the applied accelerating potential difference for which a photoelectric current due to the action of radiation on the collecting electrode could be detected, and applying to this minimum value a correction to allow for the velocity of emission of electrons from the filament, etc. The only possibility of error in the value deduced in this way is connected with the difficulty of ascertaining the appropriate correction to apply, because the special experiments made to investigate the initial energy of emission of the electrons showed that this quantity varied for different electrons over a range of about 2 volts. The determination of the minimum value of the applied potential difference at which the effects of radiation could be detected could be carried out with considerable accuracy, for the arrangement of electric fields was such as to prevent the collection of any of the original electrons, or any positive ions to which they might give rise by bombardment of the electrodes or by ionization of the gas, while, at the same time, it was such as to enable the accelerated electrons to traverse a considerable distance with their maximum energy at any applied potential difference before being retarded. It thus provided for the occurrence of collisions throughout a considerable region as soon as the applied accelerating potential difference reached a critical stage, thus causing the critical point to be well marked in the curve which showed the relation between the current measured by the electrometer and the applied accelerating potential difference.

The correction which was added to the value of the applied potential difference at the critical point, to give the energy of the electrons actually producing the effect detected, was the difference between the accelerating voltage,  $V_1$ , and the subsequent retarding voltage,  $V_2$ , when the number of electrons reaching the collecting electrode after traversing the opposing potential difference,  $V_2$ , was the minimum number required to give a measurable current. Owing to the fact that in order to reach the collecting electrode the electrons had to pass through three gauzes, the number of

electrons collected was very much smaller than the number actually having energies  $V_2 - V_1$  volts in excess of the applied accelerating potential differences. As the magnitude of the radiation current was probably somewhat similarly reduced by the intervening gauzes, the absolute value of the critical voltage deduced by the application of this maximum correction is nevertheless only likely to err on the side of being too high.

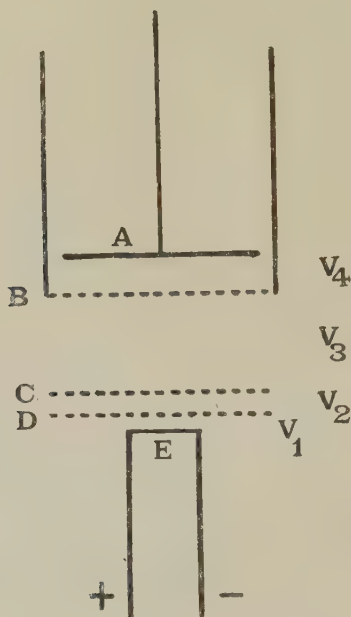
The results since obtained in an investigation of critical electron energies in neon\*, by Professor Horton and the writer, have tended to confirm the view that the correction obtained in the manner described above is too large. In the investigation referred to, the correction was determined by controlling by means of a retarding potential difference,  $V_2$ , the number of electrons which were allowed to pass into a space where they could acquire sufficient energy to make collisions of the type under investigation, and by finding the maximum value of  $V_2$  which would allow of the passage of a sufficient number of electrons to give a detectable indication of the effects of the collisions in as nearly as possible the arrangement of fields employed in obtaining the curve for which the correction was required. The values of the positive correction obtained by subtracting  $V_1$ , the first accelerating potential difference, from this value of  $V_2$ , were generally smaller than the values obtained by a method similar to that used in helium, the difference in some cases amounting to a volt or so.

The second method of ascertaining the absolute value of the critical electron energy involved the detection of the occurrence of an inelastic collision, rather than the secondary effects of an inelastic collision. It did not involve the application of a correction for energy of emission from the filament, etc. The essential features of the method can be made clear by reference to the diagram in fig. 1, in which B, C, and D, represent gauzes of fine platinum wire, B forming the base of a hollow cylindrical electrode concentric with the collecting electrode A. E denotes the source of electrons. The electrons from E were accelerated towards the gauze D by a potential difference  $V_1$  which remained constant, and those passing through this gauze were further accelerated by a potential difference  $V_2$  which could be increased in steps of one-tenth of a volt. B and C were maintained at the same potential, and the pressure used (about 1 mm. or so) was sufficiently high for all the electrons to collide with helium atoms many times before reaching B.

\* Proc. Roy. Soc. A, vol. xeviii. p. 124 (1920).

Between B and A an opposing potential difference  $V_4$  was applied which could, like  $V_2$ , be increased in steps of one tenth of a volt. Below the first critical voltage electrons make perfectly elastic collisions with helium atoms. Consequently if  $V_4$  is fixed at some value below the first critical voltage there must be a stage when, as  $V_2$  is increased, electrons are able to reach the collecting electrode and give a negative current, whereas when  $V_4$  is fixed at some value greater than the first critical voltage, inelastic collisions

Fig. 1.



must occur before the energy necessary to enable the electrons to reach the collecting electrode against  $V_4$  is attained, and if *all* the electrons make inelastic collisions, electrons from E can then never reach the collecting electrode.

The method consisted, therefore, in ascertaining for various values of  $V_4$  whether, as  $V_2$  was increased, a negative current could be obtained at any stage, and hence determining the lowest value of  $V_4$  for which no negative current could be measured as  $V_1 + V_2$  was increased to beyond the first critical stage. When  $V_1 + V_2$  exceeded a certain value

a positive current was always obtained, due presumably to the photoelectric effect of the radiation emitted at the critical stage on the collecting electrode. As the current always began to increase in the positive direction for the same value of  $V_1 + V_2$  whatever the value of  $V_4$ , the pressure must have been sufficiently high for all (or very nearly all) the electrons to have been making inelastic collisions. The lowest value of  $V_4$  at which no indication of electrons reaching the collecting electrode was obtained, was taken as the first critical electron voltage. The extreme values of the critical electron energy measured in this way were 20.3 volts and 20.45 volts, the final mean value being 20.4 volts in agreement with the value obtained by the other method.

In taking the lowest value of  $V_4$  at which there was no indication from the observations that electrons ever reached the collecting electrode, as the actual value of the first critical electron energy, it was assumed that there were no complications due to polarization layers on either of the electrodes bounding the final field. As, however, the effect of a polarization layer, if such existed, could only be to cause the measured potential difference to be smaller than the actually existing opposing potential difference, the result obtained in this way would, if it erred at all, be too low. Since the value obtained by the other method which, if it erred at all, would be too high, agreed with the value obtained by the method just described, the value 20.4 volts was taken to be the absolute value of the first critical electron energy.

The results of subsequent experiments\* have revealed the existence of two critical radiation voltages in the neighbourhood of 20 volts in helium, 0.8 volt apart. It has, moreover, been shown by Professor Horton and the writer, that the radiation current due to the recovery of the atom after the inelastic collision at the higher of the two voltages is very much larger than that which appears at the lower energy collision. This is possibly due to the fact that collisions of the higher energy type occur more readily than those of the lower energy type, but there is no definite proof of this. This possibility, combined with the fact that the current due to the secondary effects of the inelastic collision of higher energy quantum is so much greater than the current caused by collisions of the first type, raises the question as to whether the minimum value of  $V_4$  for which no indication of electrons reaching the collecting electrode

\* J. Franck and P. Knipping, *Phys. Zeits.* vol. xx. p. 481 (1919); and F. Horton and A. C. Davies, *Phil. Mag.* vol. xlii. p. 746 (1921).



was obtained in the investigation described above, corresponds to the higher energy stage, or to the stage at which the first type of inelastic collision occurs. It is possible that at the first voltage of inelastic impact some of the electrons get over without making an inelastic collision, and that the negative current due to these electrons exceeds the positive current due to the photoelectric effect of the radiation produced from those inelastic collisions which do occur, thus masking the fact that such impacts have taken place. At the second critical stage, however, fewer electrons are likely to escape making inelastic collisions, and a much increased radiation current is produced, so that it is possible that the method just described indicated the true value of the second critical stage instead of the first, as was originally thought, in which case the value of 20.4 volts is  $0.8 \pm 0.1$  volt too high for the true value of the first critical radiation voltage. As the other critical voltages determined in subsequent papers were measured relatively to the first one, a correction of the same amount must in this case be applied to them also. The correction necessary to bring the value of the ionization voltage determined experimentally into agreement with the value deduced by Professor Lyman is  $-0.7$  volt.

In a recent paper \* Franck has explained that an error of interpretation was involved in the method by which the fundamental absolute value of a critical voltage for electrons in helium was determined in Franck and Knipping's experiments. The error of interpretation in the case of their determination also was connected with the existence of the two critical energy stages 0.8 volt apart, so that the apparent difference of 0.8 volt between Lyman's calculated value and the experimentally determined value of the ionization voltage for electrons in helium, given by Franck and Knipping, is also explicable.

The bringing into alignment with the spectroscopic data of the values of the ionization voltage and some of the radiation voltages determined by other methods, does not, however, remove all the discrepancies between the results of the two lines of investigation. As has been pointed out by Lyman, there is no spectroscopic evidence of an ultra-violet line corresponding to the first radiation voltage (19.7 volts, after correction), while the faint line of doubtful origin at  $600.5 \pm 0.3$  Å.U. is the only one which approximates to correspondence with the second and most marked radiation

\* J. Franck, *Zeits. f. Phys.* vol. ii. p. 155 (1922).

voltage found by Professor Horton and the writer at 20.5 volts (after correction).

If, as Lyman supposes, the normal state of the helium atom is the state  $0S$ , and his detection of a series with variable term  $mP$  suggests that this is the case, the presence of lines  $N-1\sigma$  and  $N-1S$ , which would be those corresponding to the above-mentioned critical points, would not be in accordance with Bohr's selection principle, according to which only those transitions involving a change of one unit in the azimuthal quantum number take place\*. Experiments described by Professor Horton and the writer in the *Philosophical Magazine* for November 1921, have shown, however, that the radiations emitted at 20.4 volts and 21.2 volts (*i. e.* 19.7 volts and 20.5 volts after correction) are true resonance radiations and should, therefore, be connected by the quantum relation  $eV = h\nu$  with the critical voltages. These experiments seem to indicate that the transitions  $N-1\sigma$  and  $N-1S$  must have been occurring in the circumstances of the investigation referred to. As Franck and Knipping also obtained indisputable evidence of the emission of radiation previous to the stage which they now identify as corresponding with the emission of the line  $\lambda$  584.4, and at a voltage which would correspond with the transition  $N-1S$ , it seems that the circumstances of their investigation, and also those of the investigation of Professor Horton and the writer, must have been such as to constitute an instance when the ordinary limitations of the selection principle did not hold, and when combination series lines could be obtained.

In a recent paper † Professor Bohr, in commenting on the results of an investigation by Foote, Mohler, and Meggers, on the excitation of the potassium arc spectrum ‡, pointed out that certain results which were thought by these workers to provide an exception to the selection principle, constituted, on the contrary, evidence in favour of that principle, because the very device which was employed in order to allow recombination to occur in the absence of an applied electric field would cause accumulations of ions from which intense electric fields would result. The main objection to applying arguments of a nature similar to those put forward by Professor Bohr to the present case, lies in the fact that it has been shown that the radiations emitted at 20.4 volts

\* N. Bohr, *Phil. Mag.* vol. xliii. p. 1112 (1922).

† *Loc. cit.*

‡ P. D. Foote, F. L. Mohler, and W. F. Meggers, *Phil. Mag.* vol. xliii. p. 659 (1922).

and 21.2 volts (*i. e.* 19.7 volts and 20.5 volts after correction) can be passed on from atom to atom for a considerable distance by means of their absorption and re-emission by other helium atoms in regions where there could not possibly have been intense electric fields of the strength found by Stark \* to be essential for the emission of the lines of the series  $1S - mS$ .

The displacement of an electron to the  $1S$  orbit from the  $0S$  orbit by electron collisions would not necessarily constitute a violation of the selection principle, for it is conceivable that the close proximity of the colliding electron and the impacted atom gives rise to the necessary strong electric field. A similar argument cannot, however, be applied to the return of the electron from the  $1S$  to the  $0S$  orbit, or to the displacement of the electron to the  $1S$  orbit by the absorption of radiation. Thus the circumstances in which the radiations  $N - 1\sigma$  and  $N - 1S$  have been detected by Professor Horton and the writer seem to be such that a breaking down of the selection principle is necessarily indicated by their presence if  $0S$  is the normal state; but whether  $N$  is identical with  $0S$  or not, the fact that transitions  $N - 1S$  and  $N - 1P$  both occur must involve a breaking down of the selection principle.

Indirect evidence that the return of the displaced electron from the state  $1S$  to the normal position takes place is provided by the following considerations:—The singlet principal series of helium  $1S - mP$ , which appears in the visible and near ultra-violet part of the spectrum, corresponds to transitions in which  $1S$  is the orbit finally occupied by the moving electron, while in the case of the doublet principal series  $1\sigma$  is the orbit finally occupied. Without the absorption of further energy the atoms can then only revert to the normal state by means of the direct transitions which would give the lines  $N - 1S$  and  $N - 1\sigma$ , for there is no evidence of the transition  $1S - 1\sigma$ . Confining attention to the singlet system of lines, we see that, unless the transition  $N - 1S$  does occur, there would be, on suddenly stopping a luminous discharge, an accumulation of atoms with electrons in the  $1S$  orbits. Since the singlet principal series in the visible spectrum is comparable in intensity with the other series, the number of such abnormal atoms which would be present on stopping the discharge would be an appreciable proportion of the total number of atoms which were emitting radiation at any instant while the discharge was passing. Thus, after once having had an intense

\* J. Stark, *Ann. d. Phys.* vol. lvi. p. 577 (1918).

luminous discharge in helium, it should be possible to obtain evidence of the occurrence of ionization on raising the potential difference from zero to about 5 volts, if the transition  $N-1S$  does not occur. Such low-voltage ionization of helium does not, however, take place except when special measures are adopted to maintain a supply of abnormal atoms\*, so that there seems no alternative to supposing that the transition  $N-1S$  does occur.

The fact that this line was not clearly indicated in Professor Lyman's experiments is therefore worthy of consideration. An explanation of its absence is perhaps to be found by supposing that, although the selection principle does not hold rigorously as regards the transition  $N-1S$ , yet the transition is one which is not likely to take place when alternative displacements are possible. While a discharge is actually passing in the gas there is always the possibility of atoms with electrons in the  $1S$  states absorbing radiant energy and ultimately reverting to the normal state from one of the  $P$  states, by the transitions of whose occurrence Lyman's results give evidence. Such methods of return from the  $1S$  state to the normal state must have been possible in the circumstances of Professor Lyman's experiment, so that on the supposition made above, the line  $N-1S$  would not be expected to occur with any great intensity on Professor Lyman's plates.

In the case of the experiments of Franck and Knipping, and of Professor Horton and the writer, however, abnormal atoms with electrons in the  $1\sigma$  and  $1S$  orbits could be produced before atoms with electrons in any of the  $P$  orbits could be produced. In the interval between the voltages at which the first two types of abnormal atom were produced, and that at which electrons could be removed to the  $1P$  orbits, no radiant energy would be present unless the transitions  $N-1\sigma$  or  $N-1S$  occurred. Hence atoms with electrons in the  $1\sigma$  and the  $1S$  orbits would have no method of returning to the normal state, except by direct transitions, so that in these circumstances the radiations  $N-1\sigma$  and  $N-1S$  should be produced.

In view of the apparent discrepancies between the results of the spectroscopic and the current-voltage lines of investigation, a consideration of the extent to which current-voltage observations may be expected to reveal the existence of ultra-violet spectrum lines seems desirable. In what follows attention will be confined, for the sake of simplicity,

\* F. Horton and A. C. Davies, *Phil. Mag.* vol. xlv. p. 1140 (1922); and F. M. Kannenstine, *Astro. Phys. Journ.* vol. lv. p. 345 (1922).



to the singlet system of helium. Taking the values of the critical radiation voltages corrected to fit Professor Lyman's spectroscopic observations, it has been shown that between 20.5 volts and 21.12 volts the transition N—1S occurs, and

the current-voltage curve indicates that  $\frac{di}{de}$  increases when

the electron energy reaches 20.5 volts,  $i$  being the photo-electric current and  $e$  the applied difference of potential. At 21.12 volts electron energy the displacement of an electron to the 1P orbit can result from an encounter between a helium atom and an electron. Experiments have shown, however, that for a given number of bombarding electrons the number of inelastic collisions which occur increases as the excess of energy of the electrons above the minimum energy required for the occurrence of the inelastic collision increases. It follows, therefore, that by no means all the electronic-atomic encounters involving the necessary minimum electron energy are effective in producing electron displacements within the atom. Suppose that the probability of the displacement of an electron within a helium atom to the 1S orbit at a 21.2 volts collision is  $p_1$  and that the probability of a displacement to the 1P orbit at a 21.2 volts collision is  $p_2$ . As the transitions in question are concerned with displacements from the normal orbit to orbits of different azimuthal quantum number, it is conceivable that the circumstances necessary for the occurrence of the one are quite distinct from the circumstances necessary for the occurrence of the other. In this case the probability that one or the other will occur at 21.2 volts is  $p_1 + p_2$ . The quantity  $\frac{di}{de}$  would therefore be expected to show an increase as soon as the transition N—1P was possible.

As the applied difference of potential is increased, stages are eventually reached at which transitions to orbits 2P, 3P, etc., or orbits 2S, 3S, etc., become possible. It is difficult to conceive how, apart from the minimum voltage consideration, the circumstances necessary for the transition to the 2S orbit to take place can differ from those necessary for a displacement to the 1S orbit, and so on, or, in the case of the P terms, how the circumstances necessary for a displacement to the 2P orbit can differ from those necessary for a displacement to the 1P orbit. Hence it appears that discontinuous increases in the number of inelastic collisions occurring would not be expected to take place at the voltages corresponding to the minimum electron energies necessary to



bring about displacements to the successive orbits of higher radial quantum number than the first of any series. Discontinuous changes in  $\frac{di}{de}$  would therefore only occur at such voltages if the specific photoelectric activity of the surface acted upon by the radiation changed appreciably with wave-length in passing from  $N-mS$  to  $N-\overline{m}+1S$ , or from  $N-mP$  to  $N-\overline{m}+1P$ , and so on \*. There is no direct experimental evidence available as regards the variation of photoelectric activity with wave-length of the radiation for the region 500 Å.U. to 600 Å.U., but such a variation as that considered is very unlikely.

The arguments given above have shown that whereas the current-voltage line of investigation may reasonably be expected to give an indication of the first terms of various series corresponding to transitions involving a return of the displaced electron to the normal orbit, it by no means follows that the method will give any indication of the existence of the higher members of these series. It seems doubtful, therefore, whether Franck's interpretation of a discontinuity at a point in his curves which, after correction, becomes 22.9 volts, as corresponding to the line  $0.5S-3P$ , or, in Fowler's notation,  $0S-2P$ , is justified.

### *Summary.*

In view of the fact that there is a difference (0.7 volt) between the value of the ionization voltage for electrons in helium deduced from extreme ultra-violet spectroscopic observations by Professor Lyman, and the value obtained experimentally by Professor Horton and the writer, and independently by Franck and Knipping, the possible sources of experimental error, or error of interpretation of results, in the case of the determination made by Professor Horton and the writer are discussed.

It is shown that the discovery, made since the publication of the investigation in question, that there are *two* critical radiation voltages in the region of the first point of inelastic impact, separated by only 0.8 volt, introduces the possibility that an error of interpretation of the observed effects was made, resulting in values being attributed to all the critical

\* For the lower members of these series, the fact that in passing from, for example,  $N-1P$  to  $N-2P$ , returns to the normal orbit by the two transitions  $N-1S$  and  $1S-2P$  as an alternative to  $N-2P$  become possible, does not mean an increase of photoelectrically active radiation, for the radiations such as  $1S-2P$  would be of too long a wave-length to affect the platinum electrode.

voltages in helium which are too high by 0·8 volt  $\pm$  a small experimental error.

Evidence is produced to show that the selection principle of Bohr, according to which only those transitions occur which involve a change of one unit in the azimuthal quantum number, cannot hold rigorously with regard to transitions to the normal state in helium. A possible explanation is given of the absence of certain lines which are indicated by the current-voltage observations, from Lyman's photographic plates, and the question of the ability of the current-voltage method to reveal the existence of ultra-violet series lines is discussed. It is concluded that whereas the method may reasonably be expected to indicate the first lines of series involving a return of the displaced electron to the normal orbit, it would probably not reveal the existence of the higher members of those series.

### LXXIX. *Proceedings of Learned Societies.*

#### GEOLOGICAL SOCIETY.

[Continued from p. 384.]

December 6th, 1922.—Prof. A. C. Seward, Sc.D., F.R.S., President and afterwards Mr. R. D. Oldham, Vice-President, in the Chair.

THE following communication was read:—

‘Geological Investigations in the Falkland Islands.’ By Herbert Arthur Baker, D.Sc., D.I.C., F.G.S.

The stratigraphical succession in the Falkland Islands comprises rocks of Archæan, Devono-Carboniferous, and Permo-Carboniferous age.

There is only one exposure of Archæan rocks in the Colony: namely, in the cliffs of Cape Meredith, the southernmost point of West Falkland. The rocks seen include various igneous types (pegmatite, granite, gneiss) and metamorphic rocks (hornblende-schist, quartzite). The pegmatite and schist are also invaded by lamprophyre-dykes.

Overlying these old rocks, and separated from them by a strong unconformity, are coarse sandstones and quartzitic rocks, nearly horizontal. This unfossiliferous series is of great thickness, probably about 5000 feet. It occupies the southern part of West Falkland as far north as Port Edgar and Port Richards, and also the islands lying to the west of this area. It is generally horizontal, or dips fairly gently northwards and eastwards. It is regarded as of Devonian age, and in its lithological characters shows remarkable similarity to the Table Mountain Series of the Cape Province, with which it is correlated.

The succeeding series of rocks of Devono-Carboniferous age occupy the remainder of West Falkland (except for small areas of



Permo-Carboniferous rocks) and the northern half of East Falkland, and comprise strata correlated with the Bokkeveld and succeeding Witteberg Series of the Cape Province, to which they are notably similar in lithological character. The typical marine molluscan and crustacean fauna of the Bokkeveld Series is yielded by the middle Devonian-Carboniferous series of the Falklands, and the succeeding beds are shales and quartzites, with poorly-preserved plant-remains, like the Witteberg Beds of the Cape of Good Hope. The Middle and Upper Series each include about 2500 feet of strata.

Terrestrial deposits of Permo-Carboniferous age follow. They occupy a synclinalorium extending over the whole of the southern half of East Falkland (Lafonia) and Falkland Sound. They also occur in small, separate areas on West Falkland. The name 'Lafonian Formation' has been given to this set of beds by Dr T. G. Halle. They include a thickness of strata exceeding 9000 feet.

As in the Southern Karroo, the lowest beds of this younger system are glacial boulder-beds. There is a tillite formation, about 2300 feet thick, in every way comparable with the Dwyka Tillite. Shaly beds occur locally between the boulder-beds and the older quartzite, but are often absent. There is a thin representative of the Upper Dwyka Shales of the Southern Karroo.

A sandstone formation (Lafonian Sandstone) of no great thickness follows, and is, in turn, succeeded by more than 6000 feet of terrestrial deposits. Several thousand feet of these Upper Lafonian Beds consist of a monotonous alternation of thin sandstones and shaly beds. They have yielded the *Glossopteris* Flora in several places. These upper beds appear to be the equivalent of the Ecca Series of the Southern Karroo. In view of their thickness, it is possible that some portion of the Beaufort Series of the Karroo System may be represented, but no evidence of this was obtained.

Doleritic dykes are of frequent occurrence. Their age is post-Upper Lafonian, since they are seen to cut the highest strata in the Colony.

With regard to the precise age of the marine Devonian fauna of the fossiliferous series of the older rocks, a difficulty arises. Palæontologists in general admit no more than that this fauna is of Devonian age. Some have insisted, however, that its age is Lower Devonian. The glacial boulder-beds at the base of the Permo-Carboniferous are now very generally regarded as of Upper Carboniferous age, and the Witteberg Beds of the Cape Province have recently been considered by Sir T. W. Edgeworth David as of highest Lower Carboniferous, and possibly, in part, Middle Carboniferous age. In the Cape Province and the Falkland Islands strata of Witteberg age can be traced downwards, without a break and without traversing an excessive thickness of strata, into the marine fossiliferous series. It seems likely that the marine fauna will prove to be of Upper Devonian age.

The Falkland Islands appear to owe their existence to the fact that they occur at the crossing-place of two sets of folding movements.

LXXX. *Intelligence and Miscellaneous Articles.*

## ON THE FAILURE OF THE RECIPROCITY LAW IN PHOTOGRAPHY.

*To the Editors of the Philosophical Magazine.*

GENTLEMEN,—

I REGRET that in my paper on the Failure of the Reciprocity Law in Photography (Nov. 1922, p. 904), there is an error on page 910. It is there stated that the mean values of the constant  $p$  and the probable errors are as follows:—

For Paget "Half-Tone" plates . . . . .  $p = 0.865 \pm 0.005$   
 for Wratten "Instantaneous" plates ..  $p = 0.846 \pm 0.005$   
 for Imperial "Eclipse" plates . . . . .  $p = 0.846 \pm 0.005$

This should read:—

Paget "Half-Tone" . . . . .  $p = 0.865 \pm 0.058$   
 Wratten "Instantaneous" . . . . .  $p = 0.846 \pm 0.044$   
 Imperial "Eclipse" . . . . .  $p = 0.846 \pm 0.052$

The probable errors, as printed, are therefore incorrect. The values of  $p$  are correct.

I greatly regret that this error in computation should have occurred.

The Laboratory,  
 British Museum,  
 39 Russell Sq., W.C. 1.  
 Jan. 23, 1923.

Yours faithfully,  
 R. A. MALLET.

## PRECESSION OF THE SPINNING TOP.

*To the Editors of the Philosophical Magazine.*

GENTLEMEN,—

THE following additions should be made to my paper in the March number:—

Insert an \* after the equation in the fifth line on page 387, and add the following as a footnote:—

To cut off OL geometrically on OG to a length

$$L = \frac{OP^2}{AP^2} OA,$$

draw EL through E on PA where the angle AEO is made equal to  $\theta$ : OP is then tangent of the circle round OAE.

March 13, 1923.

Yours faithfully,  
 G. GREENHILL.